(Almost) No Label No Cry - Supplementary Material

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2 Supplementary Material on Proofs

2.1 Proof of Lemma 1

For any SPSL $F(\mathcal{S}, h)$, we can write it as ([2], Lemma 1, [3]):

$$\begin{aligned} F(\mathcal{S}, h) &= F_\phi(\mathcal{S}, h) \\ &\doteq \frac{1}{m}\sum_i D_\phi(y'_i \| \phi'^{-1}(h(\boldsymbol{x}_i))) \ , \end{aligned} \tag{1}$$

where $y'_i = 1$ iff $y_i = 1$ and 0 otherwise, ϕ is permissible and D_ϕ is the Bregman divergence with generator ϕ [3]. It also holds that: $D_\phi(y'_i \| \phi'^{-1}(h(\boldsymbol{x}_i))) = b_\phi F_\phi(yh(\boldsymbol{x}))$ with:

$$F_\phi(x) \doteq \frac{\phi^\star(-x) + \phi(0)}{\phi(0) - \phi(1/2)} = a_\phi + \frac{\phi^\star(-x)}{b_\phi} \ , \tag{2}$$

and $\phi^\star$ is the convex conjugate of ϕ, *i.e.* $\phi^\star(x) \doteq x\phi'^{-1}(x) - \phi(\phi'^{-1}(x))$. Furthermore, for any permissible ϕ, the conjex conjugate $\phi^\star(x)$ verifies the property

$$\phi^\star(-x) = \phi^\star(x) - x \ , \tag{3}$$

and so we get that:

$$\begin{aligned} F(\mathcal{S}, h) &= \frac{1}{m}\sum_i D_\phi(y'_i \| \phi'^{-1}(h(\boldsymbol{x}_i))) \\ &= \frac{b_\phi}{m}\sum_i F_\phi(y_i h(\boldsymbol{x}_i)) \\ &= \frac{b_\phi}{2m}\left(\sum_i F_\phi(y_i h(\boldsymbol{x}_i)) + \sum_i F_\phi(y_i h(\boldsymbol{x}_i))\right) \\ &= \frac{b_\phi}{2m}\left(\sum_i F_\phi(y_i h(\boldsymbol{x}_i)) + \sum_i F_\phi(-y_i h(\boldsymbol{x}_i)) - \frac{1}{b_\phi}\sum_i y_i h(\boldsymbol{x}_i)\right) \quad (4) \\ &= \frac{b_\phi}{2m}\sum_{y\in\{-1,+1\}}\sum_i F_\phi(yh(\boldsymbol{x}_i)) - \frac{1}{2m}\sum_i y_i h(\boldsymbol{x}_i) \\ &= \frac{b_\phi}{2m}\sum_{\sigma\in\{-1,+1\}}\sum_i F_\phi(\sigma h(\boldsymbol{x}_i)) - \frac{1}{2}h\left(\frac{1}{m}\sum_i y_i \boldsymbol{x}_i\right) \quad (5) \\ &= \frac{b_\phi}{2m}\sum_{\sigma\in\{-1,+1\}}\sum_i F_\phi(\sigma h(\boldsymbol{x}_i)) - \frac{1}{2}h\left(\boldsymbol{\mu}_\mathcal{S}\right) \ . \quad (6) \end{aligned}$$

(4) holds because of (3), (5) holds because h is linear. So for any samples $\mathcal{S}$ and $\mathcal{S}$ with respective size m and m', we have (again using the property that h is linear):

$$\begin{aligned} F(\mathcal{S}, h) - F(\mathcal{S}', h) &= \frac{b_\phi}{2}\sum_{\sigma\in\{-1,+1\}}\left(\frac{1}{m}\sum_{\boldsymbol{x}\in\mathcal{S}_1} F_\phi(\sigma h(\boldsymbol{x}_i)) - \frac{1}{m'}\sum_{\boldsymbol{x}\in\mathcal{S}_2} F_\phi(\sigma h(\boldsymbol{x}_i))\right) \\ &\quad + \frac{1}{2}h\left(\boldsymbol{\mu}_{\mathcal{S}_2} - \boldsymbol{\mu}_{\mathcal{S}_1}\right) \ , \end{aligned} \tag{7}$$

which yields the statement of the Lemma.

2.2 Proof of Lemma 2

Using the fact that $\mathrm{D}_{\boldsymbol{w}}$ and L are symmetric, we have:

$$\begin{aligned} &\frac{\partial \ell(\mathrm{L}, \mathrm{X})}{\partial \mathrm{X}} \\ &= -2\frac{\partial}{\partial \mathrm{X}}\mathrm{tr}\left(\mathrm{B}^\top \mathrm{D}_{\boldsymbol{w}} \Pi^\top \mathrm{X}\right) + \frac{\partial}{\partial \mathrm{X}}\mathrm{tr}\left(\mathrm{X}^\top \Pi \mathrm{D}_{\boldsymbol{w}} \Pi^\top \mathrm{X}\right) + \gamma\frac{\partial}{\partial \mathrm{X}}\mathrm{tr}\left(\mathrm{X}^\top \mathrm{L}\mathrm{X}\right) \\ &= -2\Pi\mathrm{D}_{\boldsymbol{w}}\mathrm{B} + 2\Pi\mathrm{D}_{\boldsymbol{w}}\Pi^\top \mathrm{X} + 2\gamma\mathrm{L}\mathrm{X} = 0 \ , \end{aligned}$$

out of which $\tilde{\mathrm{B}}^{\pm}$ follows in Lemma 2.

2.3 Proof of Theorem 3

We let $\Pi_o \doteq [\textsc{Diag}(\hat{\boldsymbol{\pi}})|\textsc{Diag}(\hat{\boldsymbol{\pi}}-\mathbf{1})]^\top \mathrm{N}$ an orthonormal system ($n_{jj} = (\hat{\pi}_j^2 + (1-\hat{\pi}_j)^2)^{-1/2}, \forall j \in [n]$ and 0 otherwise). Let $\mathbb{K}_{\Pi_o}$ be the n-dim subspace of $\mathbb{R}^d$ generated by Π_o. The proof of Theorem (3) exploits the following Lemma, which assumes that ε is any > 0 real for L in (8) (main file) to be $\succ 0$. When $\varepsilon = 0$, the result of Theorem (3) still holds but follows a different proof.

Lemma 1 *Let* $\mathrm{A} \doteq \Pi \mathrm{D}_{\boldsymbol{w}} \Pi^\top$ *and* L *defined as in (8) (main paper). Denote for short*

$$\mathrm{U} \;\doteq\; \left(\mathrm{L}^{-1}\mathrm{A} + \gamma^{-1}\mathrm{I}\right)^{-1} \;. \tag{8}$$

Suppose there exists $\xi > 0$ *such that for any* $\boldsymbol{x} \in \mathbb{R}^{2n}$*, the projection of* $\mathrm{U}\boldsymbol{x}$ *in* $\mathbb{K}_{\Pi_o}$*,* $\boldsymbol{x}_{U,o}$*, satisfies*

$$\|\boldsymbol{x}_{U,o}\|_2 \;\leq\; \xi\|\boldsymbol{x}\|_2 \;. \tag{9}$$

Then:

$$\|\mathrm{M} - \tilde{\mathrm{M}}\|_F \;\leq\; \gamma\xi\|\mathrm{B}^{\pm}\|_F \;. \tag{10}$$

Proof Combining Lemma 2 and (5), we get

$$\begin{aligned} \mathrm{B}^{\pm} - \tilde{\mathrm{B}}^{\pm} &= -\left(\left(\mathrm{A} + \gamma\mathrm{L}\right)^{-1}\mathrm{A} - \mathrm{I}\right)\mathrm{B}^{\pm} \\ &= \left((\gamma\mathrm{L})^{-1}\mathrm{A} + \mathrm{I}\right)^{-1}\mathrm{B}^{\pm} \;. \end{aligned} \tag{11}$$

Define the following permutation matrix:

$$\mathrm{C} \;\doteq\; \left[\begin{array}{c|c} 0 & \mathrm{I} \\ \mathrm{I} & 0 \end{array}\right] \in \mathbb{R}^{2n\times 2n} \;. \tag{12}$$

$\mathrm{A} \doteq \Pi \mathrm{D}_{\boldsymbol{w}} \Pi^\top$ is not invertible but diagonalisable. Its (orthonormal) eigenvectors can be partitioned in two matrices P_o and P such that:

$$\mathrm{P}_o \;\doteq\; [\textsc{Diag}(\hat{\boldsymbol{\pi}}-\mathbf{1})|\textsc{Diag}(\hat{\boldsymbol{\pi}})]^\top \mathrm{N} = \mathrm{C}\Pi_o \in \mathbb{R}^{2n\times n} \;\text{ (eigenvalues 0)} \;, \tag{13}$$

$$\mathrm{P} \;\doteq\; \Pi\mathrm{N} \in \mathbb{R}^{2n\times n} \;\text{ (eigenvalues } w_j(\hat{\pi}_j^2 + (1-\hat{\pi}_j)^2), \forall j) \;. \tag{14}$$

We have:

$$\begin{aligned} \mathrm{M} - \tilde{\mathrm{M}} &= \mathrm{P}_o^\top \mathrm{C}\mathrm{B}^{\pm} - \mathrm{P}_o^\top \mathrm{C}\tilde{\mathrm{B}}^{\pm} \\ &= \mathrm{P}_o^\top \mathrm{C}\left((\gamma\mathrm{L})^{-1}\mathrm{A} + \mathrm{I}\right)^{-1}\mathrm{B}^{\pm} \\ &= \Pi_o^\top \left((\gamma\mathrm{L})^{-1}\mathrm{A} + \mathrm{I}\right)^{-1}\mathrm{B}^{\pm} \qquad (15) \\ &= \gamma\Pi_o^\top \left(\mathrm{L}^{-1}\mathrm{A} + \gamma^{-1}\mathrm{I}\right)^{-1}\mathrm{B}^{\pm} \;. \qquad (16) \end{aligned}$$

Eq. (15) follows from the fact that C is idempotent. Plugging Frobenius norm in (16), we obtain

$$\begin{aligned} \|\mathrm{M} - \tilde{\mathrm{M}}\|_F^2 &= \gamma^2\|\Pi_o^\top \left(\mathrm{L}^{-1}\mathrm{A} + \gamma^{-1}\mathrm{I}\right)^{-1}\mathrm{B}^{\pm}\|_F^2 \\ &= \gamma^2 \sum_{k=1}^{d} \|\Pi_o^\top \left(\mathrm{L}^{-1}\mathrm{A} + \gamma^{-1}\mathrm{I}\right)^{-1}\boldsymbol{b}_k^{\pm}\|_2^2 \\ &\leq \gamma^2\xi^2 \sum_{k=1}^{d} \|\boldsymbol{b}_k^{\pm}\|_2^2 \qquad (17) \\ &= \gamma^2\xi^2\|\mathrm{B}^{\pm}\|_F^2 \;, \end{aligned}$$

which yields (10). In (17), $\boldsymbol{b}_k^{\pm}$ denotes *column* k in $\mathrm{B}^{\pm}$. Ineq. (17) makes use of assumption (9). ■

To ensure $\|\boldsymbol{x}_{U,o}\|_2 \leq \xi\|\boldsymbol{x}\|_2$, it is sufficient that $\|\mathrm{U}\boldsymbol{x}\|_2 \leq \xi\|\boldsymbol{x}\|_2$, and since $\|\mathrm{U}\boldsymbol{x}\|_2 \leq \|\mathrm{U}\|_F\|\boldsymbol{x}\|_2$, it is sufficient to show that

$$\left\|\mathrm{U}_\xi^{-1}\right\|_F^2 \;\leq\; 1 \;, \tag{18}$$

with $\mathrm{U}_\xi \doteq \mathrm{L}_\xi^{-1}\mathrm{A} + \xi\gamma^{-1}\mathrm{I}$, for relevant choices of ξ. We have let $\mathrm{L}_\xi \doteq (1/\xi)\mathrm{L}$. Let $0 \leq \lambda_1(.) \leq ... \leq \lambda_{2n}(.)$ denote the ordered eigenvalues of a positive-semidefinite matrix in $\mathbb{R}^{2n\times 2n}$. It follows that, since L is symmetric positive definite, we have

$$\lambda_j(\mathrm{L}_\xi^{-1}\mathrm{A}) \quad \geq \quad \frac{\lambda_j(\mathrm{A})}{\lambda_{2n}(\mathrm{L}_\xi)} \ (\geq 0) \ , \forall j \in [2n] \ .$$

We have used eq. (13). Weyl's Theorem then brings:

$$\lambda_j(\mathrm{U}_\xi^{-1}) \quad \leq \quad \frac{\lambda_{2n}(\mathrm{L}_\xi)}{\lambda_j(\mathrm{A}) + \xi\gamma^{-1}\lambda_{2n}(\mathrm{L}_\xi)} \leq \left\{ \begin{array}{ll} \xi^{-1}\gamma & \text{if} \quad j \in [n] \\ \frac{\lambda_{2n}(\mathrm{L}_\xi)}{\lambda_j(\mathrm{A})} & \text{otherwise} \end{array} \right. . \tag{19}$$

Gershgorin's Theorem brings $\lambda_{2n} \leq (1/\xi)(\varepsilon + \max_j \sum_{j'} |l_{jj'}|)$, and furthermore the eigenvalues of A satisfy $\lambda_j \geq w_j/2, \forall j \geq n+1$. We thus have:

$$\left\| \mathrm{U}_\xi^{-1} \right\|_F^2 \quad \leq \quad \frac{n\gamma^2}{\xi^2} + \frac{4n\left(\varepsilon + \max_j \sum_{j'} |l_{jj'}|\right)^2}{\xi^2 \min_j w_j^2} \ . \tag{20}$$

In (19) and (20), we have used the eigenvalues of A given in eqs (13) and (14). Assuming:

$$\gamma \quad \leq \quad \frac{\xi}{\sqrt{2n}} \ , \tag{21}$$

a sufficient condition for the right-hand side of (20) to be ≤ 1 is that

$$\xi \quad \geq \quad \frac{\varepsilon + \max_j \sum_{j'} |l_{jj'}|}{2\sqrt{n}\min_j w_j} \ . \tag{22}$$

To finish up the proof, recall that $\mathrm{L} = \mathrm{D} - \mathrm{V}$ with $d_{jj} \doteq \sum_{j,j'} v_{jj'}$ and the coordinates $v_{jj'} \geq 0$. Hence,

$$\begin{aligned} \sum_{j'} |l_{jj'}| \quad &= \quad 2\sum_{j\neq j'} v_{jj'} \\ &\leq \quad 2n \max_{j\neq j'} v_{jj'}, \forall j \in [n] \ . \end{aligned}$$

The proof is finished by plugging this upperbound in (22) to choose ξ, then taking the maximal value for γ in (21) and finally solving the upperbound in (10). This ends the proof of Theorem 3.

2.4 Proof of Lemma 4

We first consider the normalized association criterion in (10):

$$\begin{aligned} v_{jj'}^N \quad &\doteq \quad \frac{1}{2}\left(\frac{\textsc{assoc}(\mathcal{S}_j, \mathcal{S}_j)}{\textsc{assoc}(\mathcal{S}_j, \mathcal{S}_j \cup \mathcal{S}_{j'})} + \frac{\textsc{assoc}(\mathcal{S}_{j'}, \mathcal{S}_{j'})}{\textsc{assoc}(\mathcal{S}_{j'}, \mathcal{S}_j \cup \mathcal{S}_{j'})} \right) \ , \\ \textsc{assoc}(\mathcal{S}_j, \mathcal{S}_{j'}) \quad &\doteq \quad \sum_{\boldsymbol{x}\in\mathcal{S}_j, \boldsymbol{x}'\in\mathcal{S}_{j'}} \|\boldsymbol{x} - \boldsymbol{x}'\|_2^2 \ . \end{aligned} \tag{23}$$

Remark that

$$
\begin{aligned}
\|\boldsymbol{b}_j - \boldsymbol{b}_{j'}\|_2^2 &= \left\| \frac{1}{m_j} \sum_{\boldsymbol{x}_i \in \mathcal{S}_j} \boldsymbol{x}_i - \frac{1}{m_{j'}} \sum_{\boldsymbol{x}_{i'} \in \mathcal{S}_{j'}} \boldsymbol{x}_{i'} \right\|_2^2 \\
&= \frac{1}{m_j^2} \left\| \sum_{\boldsymbol{x}_i \in \mathcal{S}_j} \boldsymbol{x}_i \right\|_2^2 + \frac{1}{m_{j'}^2} \left\| \sum_{\boldsymbol{x}_{i'} \in \mathcal{S}_{j'}} \boldsymbol{x}_{i'} \right\|_2^2 - \frac{2}{m_j m_{j'}} \left(\sum_{\boldsymbol{x}_i \in \mathcal{S}_j} \boldsymbol{x}_i \right)^\top \left(\sum_{\boldsymbol{x}_{i'} \in \mathcal{S}_{j'}} \boldsymbol{x}_{i'} \right) \\
&= \frac{1}{m_j^2} \left\| \sum_{\boldsymbol{x}_i \in \mathcal{S}_j} \boldsymbol{x}_i \right\|_2^2 + \frac{1}{m_{j'}^2} \left\| \sum_{\boldsymbol{x}_{i'} \in \mathcal{S}_{j'}} \boldsymbol{x}_{i'} \right\|_2^2 - \frac{2}{m_j m_{j'}} \sum_{\boldsymbol{x}_i \in \mathcal{S}_j, \boldsymbol{x}_{i'} \in \mathcal{S}_{j'}} \boldsymbol{x}_i^\top \boldsymbol{x}_{i'} \\
&\leq \frac{1}{m_j} \sum_{\boldsymbol{x}_i \in \mathcal{S}_j} \|\boldsymbol{x}_i\|_2^2 + \frac{1}{m_{j'}} \sum_{\boldsymbol{x}_{i'} \in \mathcal{S}_{j'}} \|\boldsymbol{x}_{i'}\|_2^2 - \frac{2}{m_j m_{j'}} \sum_{\boldsymbol{x}_i \in \mathcal{S}_j, \boldsymbol{x}_{i'} \in \mathcal{S}_{j'}} \boldsymbol{x}_i^\top \boldsymbol{x}_{i'} \qquad (24) \\
&= \frac{1}{m_j m_{j'}} \sum_{\boldsymbol{x}_i \in \mathcal{S}_j, \boldsymbol{x}_{i'} \in \mathcal{S}_{j'}} \|\boldsymbol{x}_i - \boldsymbol{x}_{i'}\|_2^2 \\
&\quad \underbrace{+ \frac{m_{j'} - 1}{m_j m_{j'}} \sum_{\boldsymbol{x}_i \in \mathcal{S}_j} \|\boldsymbol{x}_i\|_2^2 + \frac{m_j - 1}{m_j m_{j'}} \sum_{\boldsymbol{x}_{i'} \in \mathcal{S}_{j'}} \|\boldsymbol{x}_{i'}\|_2^2 - \frac{1}{m_j m_{j'}} \sum_{\boldsymbol{x}_i \in \mathcal{S}_j, \boldsymbol{x}_{i'} \in \mathcal{S}_{j'}} \boldsymbol{x}_i^\top \boldsymbol{x}_{i'}}_{\doteq a} \\
&\leq \frac{2}{m_j m_{j'}} \sum_{\boldsymbol{x}_i \in \mathcal{S}_j, \boldsymbol{x}_{i'} \in \mathcal{S}_{j'}} \|\boldsymbol{x}_i - \boldsymbol{x}_{i'}\|_2^2 \qquad (25) \\
&= \frac{2}{m_j m_{j'}} \textsc{assoc}(\mathcal{S}_j, \mathcal{S}_{j'}) \ . \qquad (26)
\end{aligned}
$$

Eq. (24) exploits the fact that $\left(\sum_{j=1}^n a_j\right)^2 \leq n \left(\sum_{j=1}^n a_j^2\right)$ and eq. (25) exploits the fact that $a \leq (m_j m_{j'})^{-1} \sum_{\boldsymbol{x}_i \in \mathcal{S}_j, \boldsymbol{x}_{i'} \in \mathcal{S}_{j'}} \|\boldsymbol{x}_i - \boldsymbol{x}_{i'}\|_2^2$. We thus have:

$$
\begin{aligned}
\frac{\textsc{assoc}(\mathcal{S}_j, \mathcal{S}_j)}{\textsc{assoc}(\mathcal{S}_j, \mathcal{S}_j \cup \mathcal{S}_{j'})} &= \frac{\textsc{assoc}(\mathcal{S}_j, \mathcal{S}_j)}{\textsc{assoc}(\mathcal{S}_j, \mathcal{S}_j) + \textsc{assoc}(\mathcal{S}_j, \mathcal{S}_{j'})} \\
&\leq \frac{\textsc{assoc}(\mathcal{S}_j, \mathcal{S}_j)}{\textsc{assoc}(\mathcal{S}_j, \mathcal{S}_j) + \frac{m_j m_{j'}}{2} \|\boldsymbol{b}_j - \boldsymbol{b}_{j'}\|_2^2} \qquad (27) \\
&\leq \frac{\kappa' m_j}{\kappa' m_j + \frac{m_j m_{j'}}{2} \|\boldsymbol{b}_j - \boldsymbol{b}_{j'}\|_2^2} \qquad (28) \\
&= \frac{1}{1 + \frac{m_{j'}}{2\kappa'} \|\boldsymbol{b}_j - \boldsymbol{b}_{j'}\|_2^2} \ . \qquad (29)
\end{aligned}
$$

Eq. (27) uses (26) and eq. (28) uses assumption (**D2**). Eq. (28) also holds when permuting j and j', so we get:

$$\begin{aligned}
\varsigma(\mathrm{V}^{NC}, \mathrm{B}^{\pm}) &\leq \max_{j \neq j'} \left(\frac{\varepsilon}{2n} + \frac{1}{1 + \frac{m_j}{2\kappa'} \|\boldsymbol{b}_j - \boldsymbol{b}_{j'}\|_2^2} + \frac{1}{1 + \frac{m_{j'}}{2\kappa'} \|\boldsymbol{b}_j - \boldsymbol{b}_{j'}\|_2^2} \right)^2 \|\mathrm{B}^{\pm}\|_F \\
&\leq \left(\frac{\varepsilon}{2n} + \frac{1}{1 + \frac{\min_j m_j}{2\kappa'} \min_{j,j'} \|\boldsymbol{b}_j - \boldsymbol{b}_{j'}\|_2^2} \right)^2 \|\mathrm{B}^{\pm}\|_F \\
&\leq \left(\frac{\varepsilon^2}{2n^2} + 2 \left(\frac{1}{1 + \frac{\min_j m_j}{2\kappa'} \min_{j,j'} \|\boldsymbol{b}_j - \boldsymbol{b}_{j'}\|_2^2} \right)^2 \right) \|\mathrm{B}^{\pm}\|_F \qquad (30) \\
&\leq \frac{\varepsilon^2}{2n^2} d \max_{\sigma,j} \|\boldsymbol{b}_j^{\sigma}\|_2 + \frac{4\kappa' d \max_{\sigma,j} \|\boldsymbol{b}_j^{\sigma}\|_2}{\min_{j,j'}^2 \|\boldsymbol{b}_j - \boldsymbol{b}_{j'}\|_2^2} \\
&\leq \frac{\varepsilon^2}{2n^2} d \max_{\sigma,j} \|\boldsymbol{b}_j^{\sigma}\|_2 + \frac{4\kappa' d}{\kappa^2 \max_{\sigma,j} \|\boldsymbol{b}_j^{\sigma}\|_2} \\
&= f^{NC}\left(\max_{\sigma,j} \|\boldsymbol{b}_j^{\sigma}\|_2 \right) \\
&= o(1) \ , \qquad (31)
\end{aligned}$$

where the last inequality uses assumption (**D1**), and (30) uses the property that $(a+b)^2 \leq 2a^2 + 2b^2$. We have let

$$f^{NC}(x) \doteq \frac{\varepsilon^2}{2n^2} dx + \frac{4\kappa' d}{\kappa x} \ , \qquad (32)$$

which is indeed $o(1)$ if $\varepsilon = o(n^2/\sqrt{x})$. This proves the Lemma for $\varsigma(\mathrm{V}^{NC}, \mathrm{B}^{\pm})$. The case of $\varsigma(\mathrm{V}^{G,s}, \mathrm{B}^{\pm})$ is easier, as

$$\begin{aligned}
\exp\left(-\frac{\|\boldsymbol{b}_j - \boldsymbol{b}_{j'}\|_2}{s} \right) &\leq \exp\left(-\frac{\min_{j'',j'''} \|\boldsymbol{b}_{j''} - \boldsymbol{b}_{j'''}\|_2}{s} \right) \\
&\leq \exp\left(-\frac{\kappa}{s} \max_{\sigma,j} \|\boldsymbol{b}_j^{\sigma}\|_2 \right) \ ,
\end{aligned}$$

from assumption (**D1**) alone, which gives

$$\begin{aligned}
\varsigma(\mathrm{V}^{G,s}, \mathrm{B}^{\pm}) &\leq \|\mathrm{B}^{\pm}\|_F \left(\frac{\varepsilon}{2n} + \exp\left(-\frac{\kappa}{s} \max_{\sigma,j} \|\boldsymbol{b}_j^{\sigma}\|_2 \right) \right)^2 \\
&\leq \|\mathrm{B}^{\pm}\|_F \left(\frac{\varepsilon^2}{2n^2} + 2\exp\left(-\frac{2\kappa}{s} \max_{\sigma,j} \|\boldsymbol{b}_j^{\sigma}\|_2 \right) \right) \\
&\leq d \max_{\sigma,j} \|\boldsymbol{b}_j^{\sigma}\|_2 \left(\frac{\varepsilon^2}{2n^2} + 2\exp\left(-\frac{2\kappa}{s} \max_{\sigma,j} \|\boldsymbol{b}_j^{\sigma}\|_2 \right) \right) \\
&= f^{G}\left(\max_{\sigma,j} \|\boldsymbol{b}_j^{\sigma}\|_2 \right) \\
&= o(1) \ , \qquad (33)
\end{aligned}$$

as claimed. We have let $f^G(x) \doteq \frac{\varepsilon^2}{2n^2} dx + dx \exp(-2\kappa x/s)$, which is indeed $o(1)$ if $\varepsilon = o(n^2/\sqrt{x})$. Remark that we shall have in general $f^G(x) \leq f^{NC}(x)$ and even $f^G(x) = o(f^{NC}(x))$ if $\varepsilon = 0$, so we may expect better convergence in the case of $\mathrm{V}^{G,s}$ as $\max_{\sigma,j} \|\boldsymbol{b}_j^{\sigma}\|_2$ grows.

2.5 Proof of Lemma 5

We first restate the Lemma in a more explicit way, that shall provide explicit values for κ_l and κ_n.

Lemma 2 *There exist $\kappa_{jj'}$ and $s_{jj'}$ depending on $d_j, d_{j'}$, and $\kappa'_{jj'} > 1$ depending on $m_j, m_{j'}$, such that:*

- *If* $v_{jj'}^{G,s_{jj'}} > \exp(-1/4)$ *then* $\mathcal{S}_j, \mathcal{S}_{j'}$ *are not linearly separable;*
- *If* $v_{jj'}^{G,s_{jj'}} < \exp(-64)$ *then* $\mathcal{S}_j, \mathcal{S}_{j'}$ *are linearly separable;*
- *If* $v_{jj'}^{NC} > \kappa_{jj'}$ *then* $\mathcal{S}_j, \mathcal{S}_{j'}$ *are not linearly separable;*
- *If* $v_{jj'}^{NC} < \kappa_{jj'}/\kappa'_{jj'}$ *then* $\mathcal{S}_j, \mathcal{S}_{j'}$ *are linearly separable.*

Proof We first consider the normalized association criterion in (10), and we prove the Lemma for the following expressions of $\kappa_{jj'}$ and $\kappa'_{jj'}$:

$$\kappa_{jj'} \doteq \frac{16}{2+\frac{d_{jj'}^2}{2d_{j'}^2}} + \frac{16}{2+\frac{d_{jj'}^2}{2d_j^2}} , \tag{34}$$

$$\kappa'_{jj'} \doteq 512\max\{m_j, m_{j'}\} , \tag{35}$$

with $d_{jj'} \doteq \max\{d_j, d_{j'}\}$ and $d_j \doteq \max_{\boldsymbol{x},\boldsymbol{x}' \in \mathcal{S}_j} \|\boldsymbol{x}-\boldsymbol{x}'\|_2$, $\forall j \neq j' \in [n]$. For any bag $\mathcal{S}_j$, we let $(\boldsymbol{b}_j^\star, r_j) \doteq MEB(\mathcal{S}_j)$ denote the minimum enclosing ball (MEB) for bag $\mathcal{S}_j$ and distance L_2, that is, r_j is the smallest unique real such that

$$\exists! \boldsymbol{b}_j^\star : d(\boldsymbol{x}, \boldsymbol{b}_j^\star) \doteq \|\boldsymbol{x}-\boldsymbol{b}_j^\star\|_2 \leq r_j, \forall \boldsymbol{x} \in \mathcal{S}_j .$$

We have let $d(\boldsymbol{x}, \boldsymbol{b}_j^\star) \doteq \|\boldsymbol{x}-\boldsymbol{b}_j^\star\|_2$. We are going to prove a first result involving the MEBs of $\mathcal{S}_j$ and $\mathcal{S}_{j'}$, and then will translate the result to the Lemma's statement. The following properties follows from standard properties of MEBs and the fact that $d(.,.)$ is a distance (they hold for any $j \neq j'$):

(a) $d(\boldsymbol{x}, \boldsymbol{x}') \leq 2r_j \ , \forall \boldsymbol{x}, \boldsymbol{x}' \in \mathcal{S}_j$;

(b) If bags $\mathcal{S}_j$ and $\mathcal{S}_{j'}$ are linearly separable, then $\forall \boldsymbol{x} \in \textsc{co}(\mathcal{S}_j)$, $\exists \boldsymbol{x}' \in \mathcal{S}_{j'}$ such that $d(\boldsymbol{x}, \boldsymbol{x}') \geq \max\{r_j, r_{j'}\}$; here, "CO" denotes the convex closure;

(c) If bags $\mathcal{S}_j$ and $\mathcal{S}_{j'}$ are linearly separable, then $d(\boldsymbol{b}_j, \boldsymbol{b}_{j'}) \geq \max\{r_j, r_{j'}\}$, where $\boldsymbol{b}_j$ and $\boldsymbol{b}_{j'}$ are the bags average;

(d) $\forall \boldsymbol{x} \in \mathcal{S}_j, \exists \boldsymbol{x}' \in \mathcal{S}_j$ s.t. $d(\boldsymbol{x}, \boldsymbol{x}') \geq r_j$;

(e) $d(\boldsymbol{x}, \boldsymbol{x}') \leq 2\max\{r_j, r_{j'}\} + d(\boldsymbol{b}_j^\star, \boldsymbol{b}_{j'}^\star), \forall \boldsymbol{x} \in \textsc{co}(\mathcal{S}_j), \forall \boldsymbol{x}' \in \textsc{co}(\mathcal{S}_{j'})$.

Let us define

$$\textsc{assoc}(\mathcal{S}_j, \mathcal{S}_{j'}) \doteq \sum_{\boldsymbol{x} \in \mathcal{S}_j, \boldsymbol{x}' \in \mathcal{S}_{j'}} d^2(\boldsymbol{x}, \boldsymbol{x}') . \tag{36}$$

We remark that, assuming that each bag contains at least two elements without loss of generality:

$$v_{jj'}^{NC} = \frac{1}{2}\left(\frac{1}{1+\frac{\textsc{assoc}(\mathcal{B}_j, \mathcal{B}_{j'})}{\textsc{assoc}(\mathcal{B}_j, \mathcal{B}_j)}} + \frac{1}{1+\frac{\textsc{assoc}(\mathcal{B}_j, \mathcal{B}_{j'})}{\textsc{assoc}(\mathcal{B}_{j'}, \mathcal{B}_{j'})}}\right) . \tag{37}$$

We have $\textsc{assoc}(\mathcal{S}_j, \mathcal{S}_j) \leq 4m_j r_j^2$ and $\textsc{assoc}(\mathcal{S}_{j'}, \mathcal{S}_{j'}) \leq 4m_{j'} r_{j'}^2$ (because of (a)), and also $\textsc{assoc}(\mathcal{S}_j, \mathcal{S}_{j'}) \geq \max\{m_j, m_{j'}\}\max\{r_j^2, r_{j'}^2\}$ when $\mathcal{S}_j$ and $\mathcal{S}_{j'}$ are linearly separable (because of (b)), which yields in this case

$$\begin{aligned} v_{jj'}^{NC} &\leq \frac{1}{2+\frac{\max\{m_j,m_{j'}\}\max\{r_j^2,r_{j'}^2\}}{2m_j r_j^2}} + \frac{1}{2+\frac{\max\{m_j,m_{j'}\}\max\{r_j^2,r_{j'}^2\}}{2m_{j'} r_{j'}^2}} \\ &\leq \frac{1}{2+\frac{\max\{r_j^2,r_{j'}^2\}}{2r_j^2}} + \frac{1}{2+\frac{\max\{r_j^2,r_{j'}^2\}}{2r_{j'}^2}} . \end{aligned} \tag{38}$$

Let us name $\kappa_{jj'}^\circ$ the right-hand side of (38). It follows that when $v_{jj'}^{NC} > \kappa_{jj'}^\circ$, $\mathcal{S}_j$ and $\mathcal{S}_{j'}$ are not linearly separable.

On the other hand, we have $\textsc{assoc}(\mathcal{S}_j, \mathcal{S}_j) \geq m_j r_j^2$ and $\textsc{assoc}(\mathcal{S}_{j'}, \mathcal{S}_{j'}) \geq m_{j'} r_{j'}^2$ (because of (d)), and also

$$\begin{aligned} \textsc{assoc}(\mathcal{S}_j, \mathcal{S}_{j'}) &\leq m_j m_{j'} (2\max\{r_j, r_{j'}\} + d(\boldsymbol{b}_j^\star, \boldsymbol{b}_{j'}^\star))^2 \\ &\leq m_j m_{j'} (4\max\{r_j^2, r_{j'}^2\} + 2d^2(\boldsymbol{b}_j^\star, \boldsymbol{b}_{j'}^\star)) \ , \end{aligned} \tag{39}$$

because of (e) and the fact that $(a+b)^2 \leq 2a^2 + 2b^2$. It follows that $\forall j \neq j'$:

$$v_{jj'}^{NC} \geq \frac{1}{2 + \frac{2m_{j'}(4\max\{r_j^2, r_{j'}^2\} + 2d^2(\boldsymbol{b}_j^\star, \boldsymbol{b}_{j'}^\star))}{r_j^2}} + \frac{1}{2 + \frac{2m_j(4\max\{r_j^2, r_{j'}^2\} + 2d^2(\boldsymbol{b}_j^\star, \boldsymbol{b}_{j'}^\star))}{r_{j'}^2}} \ . \tag{40}$$

For any $j \neq j'$, when $d^2(\boldsymbol{b}_j^\star, \boldsymbol{b}_{j'}^\star) \leq 4\max\{r_j^2, r_{j'}^2\}$, then we have from (40):

$$\begin{aligned} v_{jj'}^{NC} &\geq \frac{1}{2 + \frac{16 m_{j'} \max\{r_j^2, r_{j'}^2\}}{r_j^2}} + \frac{1}{2 + \frac{16 m_j \max\{r_j^2, r_{j'}^2\}}{r_{j'}^2}} \\ &> \kappa_{jj'}^\circ / (32\max\{m_j, m_{j'}\}) \ . \end{aligned} \tag{41}$$

Hence, when $v_{jj'}^{NC} \leq \kappa_{jj'}^\circ / (32\max\{m_j, m_{j'}\})$, it implies $d(\boldsymbol{b}_j^\star, \boldsymbol{b}_{j'}^\star) > 2\max\{r_j, r_{j'}\}$, implying $d(\boldsymbol{b}_j^\star, \boldsymbol{b}_{j'}^\star) > r_j + r_{j'}$, which is a sufficient condition for the linear separability of $\mathcal{S}_j$ and $\mathcal{S}_{j'}$.

So, we can relate the linear separability of $\mathcal{S}_j$ and $\mathcal{S}_{j'}$ to the value of $v_{jj'}^{NC}$ with respect to $\kappa_{jj'}^\circ$ defined in (38). To remove the dependence in the MEB parameters and obtain the statement of the Lemma, we just have to remark that $d_j^2/4 \leq r_j^2 \leq 4d_j^2, \forall j \in [n]$, which yields $\kappa_{jj'}/16 \leq \kappa_{jj'}^\circ \leq \kappa_{jj'}$. Hence, when $v_{jj'}^{NC} > \kappa_{jj'}$, it follows that $v_{jj'}^{NC} > \kappa_{jj'}^\circ$ and $\mathcal{S}_j$ and $\mathcal{S}_{j'}$ are not linearly separable. On the other hand, when $v_{jj'}^{NC} \leq \kappa_{jj'}/(16 \times 32\max\{m_j, m_{j'}\}) = \kappa_{jj'}/\kappa'_{jj'}$, then $v_{jj'}^{NC} \leq \kappa_{jj'}^\circ / (32\max\{m_j, m_{j'}\})$ and the bags $\mathcal{S}_j$ and $\mathcal{S}_{j'}$ are linearly separable. This achieves the proof of Lemma 5 for the normalized association criterion in (10).

The proof for $v_{jj'}^{G,s}$ is shorter, and we prove it for

$$s_{j,j'} = \max\{d_j, d_{j'}\} \ . \tag{42}$$

We have $(1/2)\max\{d_j, d_{j'}\} \leq \max\{r_j, r_{j'}\} \leq 2\max\{d_j, d_{j'}\}$. Hence, because of (c) above, if $\mathcal{S}_j$ and $\mathcal{S}_{j'}$ are linearly separable, then $v_{jj'}^{G,s} \leq 1/e^{1/4}$; so, when $v_{jj'}^{G,s} > 1/e^{1/4}$, the two bags are not linearly separable. On the other hand, if $d(\boldsymbol{b}_j^\star, \boldsymbol{b}_{j'}^\star) \leq 2\max\{r_j, r_{j'}\}$, then because of (e) above $d(\boldsymbol{b}_j, \boldsymbol{b}_{j'}) \leq 4\max\{r_j, r_{j'}\} \leq 8\max\{d_j, d_{j'}\}$, and so $v_{jj'}^{G,s} \geq 1/e^{64}$. This implies that if $v_{jj'}^{G,s} < 1/e^{64}$, then $d(\boldsymbol{b}_j^\star, \boldsymbol{b}_{j'}^\star) > 2\max\{r_j, r_{j'}\} \geq r_j + r_{j'}$, and thus the two bags are linearly separable, as claimed.

This achieves the proof of Lemma 2. ■

This achieves the proof of Lemma 5.

2.6 Mean Map estimator's Lemma and Proof

It is not hard to check that the randomized procedure that builds $\tilde{\mu}_{\mathcal{S}}^{\text{RAND}} \doteq y\boldsymbol{x}$ for some random $\boldsymbol{x} \in \mathcal{S}$ and $y \in \{-1, 1\}$ guarantees $O(2+\gamma)$ approximability when some bags are close to the convex hull of $\mathcal{S}$, for small $\gamma > 0$. Hence, the Mean Map estimation of $\boldsymbol{\mu}_{\mathcal{S}}$ can be very poor in that respect.

Lemma 3 *For any $\gamma > 0$, the Mean Map estimator $\tilde{\mu}_{\mathcal{S}}^{\text{MM}}$ cannot guarantee $\|\tilde{\boldsymbol{\mu}}_{\mathcal{S}}^{\text{MM}} - \boldsymbol{\mu}_{\mathcal{S}}\|_2 / \max_{\sigma,j} \|\boldsymbol{b}_j^\sigma\|_2 \leq 2 - \gamma$, even when (**D1 + D2**) hold.*

Proof Let $x > 0, \epsilon \in (0,1), p \in (0,1), p \neq 1/2$. We create a dataset from four observations, $\{(x_1 = 0, 1), (x_2 = 0, -1), (x_3 = x, 1), (x_4 = x, -1)\}$. There are two bags, $\mathcal{S}_1$ takes $1-\epsilon$ of x_2 and ϵ of x_1. $\mathcal{S}_2$ takes ϵ of x_4 and $1-\epsilon$ of x_3. The label-wise estimators $\tilde{\mu}^\sigma$ of [4] are solution of

$$\begin{aligned} \begin{bmatrix} \tilde{\mu}^1 \\ \tilde{\mu}^{-1} \end{bmatrix} &= \left(\begin{bmatrix} 1-\epsilon & \epsilon \\ \epsilon & 1-\epsilon \end{bmatrix}^\top \begin{bmatrix} 1-\epsilon & \epsilon \\ \epsilon & 1-\epsilon \end{bmatrix} \right)^{-1} \begin{bmatrix} 1-\epsilon & \epsilon \\ \epsilon & 1-\epsilon \end{bmatrix}^\top \begin{bmatrix} x \\ 0 \end{bmatrix} \\ &= \frac{1}{1-2\epsilon} \begin{bmatrix} (1-\epsilon)x \\ \epsilon x \end{bmatrix} \end{aligned} \tag{43}$$

On the other hand, the true quantities are:

$$\begin{bmatrix} \mu^1 \\ \mu^{-1} \end{bmatrix} = \begin{bmatrix} (1-\epsilon)x \\ \epsilon x \end{bmatrix} . \quad (44)$$

We now mix classes in $\mathcal{S}$ and pick bag proportions $q \doteq \mathbb{P}_{\mathcal{S}}[\mathcal{S}_1]$ and $1-q = \mathbb{P}_{\mathcal{S}}[\mathcal{S}_2]$. We have the class proportions defined by $\mathbb{P}_{\mathcal{S}}[y=+1] = \epsilon q + (1-\epsilon)(1-q) \doteq p$. Then

$$\begin{aligned} |\tilde{\mu}_{\mathcal{S}} - \mu_{\mathcal{S}}| &= \left| p(1-\epsilon)\left(\frac{1}{1-2\epsilon} - 1\right)x - (1-p)\epsilon\left(\frac{1}{1-2\epsilon} - 1\right)x \right| \\ &= \frac{2\epsilon|p-\epsilon|}{1-2\epsilon}x \\ &= 2\epsilon(1-q)x . \end{aligned} \quad (45)$$

Furthermore, $\max_i |b_i^\sigma| = x$. We get

$$\frac{|\tilde{\mu}_{\mathcal{S}} - \mu_{\mathcal{S}}|}{\max_i |b_i^\sigma|} = 2\epsilon(1-q) . \quad (46)$$

Picking ϵ and $(1-q)$ both $> \sqrt{1-(\gamma/2)}$ is sufficient to have eq. (46) $> 2-\gamma$ for any $\gamma > 0$. Remark that both assumptions (**D1**) and (**D2**) hold for any $\kappa < 1$ and any $\kappa' > 0$. ■

2.7 Proof of Theorem 6

The proof of the Theorem involves two Lemmata, the first of which is of independent interest and holds for any convex twice differentiable function F, and not just any F_ϕ. So, let us define:

$$F(\mathcal{S}_{|y}, \boldsymbol{\theta}, \boldsymbol{\mu}) = \frac{b}{2m}\left(\sum_i \sum_\sigma F(\sigma \boldsymbol{\theta}^\top \boldsymbol{x}_i)\right) - \frac{1}{2}\boldsymbol{\theta}^\top \boldsymbol{\mu} . \quad (47)$$

where b is any fixed positive real. Define also the regularized loss:

$$F(\mathcal{S}_{|y}, \boldsymbol{\theta}, \boldsymbol{\mu}, \lambda) \doteq F(\mathcal{S}_{|y}, \boldsymbol{\theta}, \boldsymbol{\mu}) + \lambda \|\boldsymbol{\theta}\|_2^2 . \quad (48)$$

Let $\boldsymbol{f}_k \in \mathbb{R}^m$ denote the vector encoding the k^{th} variable in $\mathcal{S}$: $f_{ki} = x_{ik}$. For any $k \in [d]$, let

$$\tilde{\boldsymbol{f}}_k \doteq \left(\frac{d}{\sum_k \|\boldsymbol{f}_k\|_2^2}\right)^{\frac{d-1}{2d}} \boldsymbol{f}_k \quad (49)$$

denote a normalization of vectors $\boldsymbol{f}_k$ in the sense that

$$\begin{aligned} \frac{1}{d}\sum_k \|\tilde{\boldsymbol{f}}_k\|_2^2 &= \frac{1}{d}\left(\frac{d}{\sum_k \|\boldsymbol{f}_k\|_2^2}\right)^{1-\frac{1}{d}} \sum_k \|\boldsymbol{f}_k\|_2^2 \\ &= \left(\frac{1}{d}\sum_k \|\boldsymbol{f}_k\|_2^2\right)^{\frac{1}{d}} . \end{aligned} \quad (50)$$

Let $\tilde{\mathrm{V}}$ collect all vectors $\tilde{\boldsymbol{f}}_k$ in column and V collect all vectors $\boldsymbol{f}_k$ in column. Without loss of generality, we assume $\mathrm{V}^\top \mathrm{V} \succ 0$, *i.e.* $\mathrm{V}^\top \mathrm{V}$ positive definite (*i.e.* no feature is a linear combination of the others), implying, because the columns of $\tilde{\mathrm{V}}$ are just positive rescaling of the columns of V, that $\tilde{\mathrm{V}}^\top \tilde{\mathrm{V}} \succ 0$ as well. We use V instead of F as in the main paper, in order not to counfound with the general convex surrogate notation F that we use here.

Lemma 4 *Given any two $\boldsymbol{\mu}$ and $\boldsymbol{\mu}'$, let $\boldsymbol{\theta}_*$ and $\boldsymbol{\theta}'_*$ be the respective minimizers of $F(\mathcal{S}_{|y}, ., \boldsymbol{\mu}, \lambda)$ and $F(\mathcal{S}_{|y}, ., \boldsymbol{\mu}', \lambda)$. Suppose there exists $F''_\circ > 0$ such that surrogate F satisfies*

$$F''(\pm(\alpha\boldsymbol{\theta}_* + (1-\alpha)\boldsymbol{\theta}'_*)^\top \boldsymbol{x}_i) \geq F''_\circ , \forall \alpha \in [0,1], \forall i \in [m] . \quad (51)$$

Then the following holds:

$$\|\boldsymbol{\theta}_* - \boldsymbol{\theta}'_*\|_2 \leq \frac{1}{2\lambda + \frac{2}{em}F''_\circ \mathrm{vol}^2(\tilde{\mathrm{V}})} \|\boldsymbol{\mu} - \boldsymbol{\mu}'\|_2 , \quad (52)$$

where $\mathrm{vol}(\tilde{\mathrm{V}}) \doteq \sqrt{\det \tilde{\mathrm{V}}^\top \tilde{\mathrm{V}}}$ *denote the volume of the (row/column) system of* $\tilde{\mathrm{V}}$.

Proof Our proof begins following the same first steps as the proof of Lemma 17 in [5], adding the steps that handle the lowerbound on F''. Consider the following auxiliary function $A_F(\boldsymbol{\tau})$:

$$A_F(\boldsymbol{\tau}) \doteq \left(\nabla F(\mathcal{S}_{|y}, \boldsymbol{\theta}_*, \boldsymbol{\mu}) - \nabla F(\mathcal{S}_{|y}, \boldsymbol{\theta}'_*, \boldsymbol{\mu}')\right)^\top (\boldsymbol{\tau} - \boldsymbol{\theta}'_*) + \lambda \|\boldsymbol{\tau} - \boldsymbol{\theta}'_*\|_2^2 \ , \tag{53}$$

where the gradient ∇ of F is computed with respect to parameter $\boldsymbol{\theta}$. The gradient of $A_F(.)$ is:

$$\nabla A_F(\boldsymbol{\tau}) = \nabla F(\mathcal{S}_{|y}, \boldsymbol{\theta}_*, \boldsymbol{\mu}) - \nabla F(\mathcal{S}_{|y}, \boldsymbol{\theta}'_*, \boldsymbol{\mu}') + 2\lambda(\boldsymbol{\tau} - \boldsymbol{\theta}'_*) \ , \tag{54}$$

The gradient of A_F satisfies

$$\begin{aligned} \nabla A_F(\boldsymbol{\theta}_*) &= \nabla F(\mathcal{S}_{|y}, \boldsymbol{\theta}_*, \boldsymbol{\mu}, \lambda) - \nabla F(\mathcal{S}_{|y}, \boldsymbol{\theta}'_*, \boldsymbol{\mu}', \lambda) \\ &= \mathbf{0} \ , \end{aligned} \tag{55}$$

as both gradients in the right are $\mathbf{0}$ because of the optimality of $\boldsymbol{\theta}_*$ and $\boldsymbol{\theta}'_*$ with respect to $F(\mathcal{S}_{|y}, ., \boldsymbol{\mu}, \lambda)$ and $F(\mathcal{S}_{|y}, ., \boldsymbol{\mu}', \lambda)$. The Hessian H of A_F is $\mathrm{H}A_F(\boldsymbol{\tau}) = 2\lambda \mathrm{I} \succeq 0$ and so A_F is convex and is thus minimal at $\boldsymbol{\tau} = \boldsymbol{\theta}_*$. Finally, $A_F(\boldsymbol{\theta}'_*) = 0$. It comes thus $A_F(\boldsymbol{\theta}_*) \leq 0$, which yields equivalently:

$$\begin{aligned} 0 \geq & \left(\nabla F(\mathcal{S}_{|y}, \boldsymbol{\theta}_*, \boldsymbol{\mu}) - \nabla F(\mathcal{S}_{|y}, \boldsymbol{\theta}'_*, \boldsymbol{\mu}')\right)^\top (\boldsymbol{\theta}_* - \boldsymbol{\theta}'_*) + \lambda \|\boldsymbol{\theta}_* - \boldsymbol{\theta}'_*\|_2^2 \\ &= \left(\frac{b}{2m} \sum_y \sum_i \nabla F(y \boldsymbol{\theta}_*^\top \boldsymbol{x}_i) - \frac{1}{2}\boldsymbol{\mu} - \frac{b}{2m} \sum_y \sum_i \nabla F(y \boldsymbol{\theta}_*'^\top \boldsymbol{x}_i) + \frac{1}{2}\boldsymbol{\mu}'\right)^\top (\boldsymbol{\theta}_* - \boldsymbol{\theta}'_*) \\ &+ \lambda \|\boldsymbol{\theta}_* - \boldsymbol{\theta}'_*\|_2^2 \\ = & \underbrace{\frac{b}{2m} \left(\sum_y \sum_i \nabla F(y \boldsymbol{\theta}_*^\top \boldsymbol{x}_i) - \sum_y \sum_i \nabla F(y \boldsymbol{\theta}_*'^\top \boldsymbol{x}_i)\right)^\top (\boldsymbol{\theta}_* - \boldsymbol{\theta}'_*)}_{\doteq a} \\ &- \frac{1}{2} (\boldsymbol{\mu} - \boldsymbol{\mu}')^\top (\boldsymbol{\theta}_* - \boldsymbol{\theta}'_*) + \lambda \|\boldsymbol{\theta}_* - \boldsymbol{\theta}'_*\|_2^2 \ . \end{aligned} \tag{56}$$

Let us lowerbound a. We have $\nabla F(y \boldsymbol{\theta}_*^\top \boldsymbol{x}) = y F'(y \boldsymbol{\theta}_*^\top \boldsymbol{x}) \boldsymbol{x}$, and a Taylor expansion brings that for any $\boldsymbol{\theta}_*, \boldsymbol{\theta}'_*$, there exists some $\alpha \in [0, 1]$ such that, defining

$$u_{\alpha,i} \doteq y(\alpha \boldsymbol{\theta}_* + (1 - \alpha) \boldsymbol{\theta}'_*)^\top \boldsymbol{x}_i \ , \tag{57}$$

we have:

$$F'(y \boldsymbol{\theta}_*^\top \boldsymbol{x}_i) = F'(y \boldsymbol{\theta}_*'^\top \boldsymbol{x}_i) + y(\boldsymbol{\theta}_* - \boldsymbol{\theta}'_*)^\top \boldsymbol{x}_i F''(u_{\alpha,i}) \ . \tag{58}$$

We thus get:

$$\begin{aligned} a &= \left(\sum_y \sum_i \nabla F(y \boldsymbol{\theta}_*^\top \boldsymbol{x}_i) - \sum_y \sum_i \nabla F(y \boldsymbol{\theta}_*'^\top \boldsymbol{x}_i)\right)^\top (\boldsymbol{\theta}_* - \boldsymbol{\theta}'_*) \\ &= \left(\sum_y \sum_i y(F'(y \boldsymbol{\theta}_*^\top \boldsymbol{x}_i) - F'(y \boldsymbol{\theta}_*'^\top \boldsymbol{x}_i)) \boldsymbol{x}_i\right)^\top (\boldsymbol{\theta}_* - \boldsymbol{\theta}'_*) \\ &= \left(\sum_y \sum_i (\boldsymbol{\theta}_* - \boldsymbol{\theta}'_*)^\top \boldsymbol{x}_i F''(u_{\alpha,i}) \boldsymbol{x}_i\right)^\top (\boldsymbol{\theta}_* - \boldsymbol{\theta}'_*) \\ &= 2 \sum_i ((\boldsymbol{\theta}_* - \boldsymbol{\theta}'_*)^\top \boldsymbol{x}_i)^2 F''(u_{\alpha,i}) \\ &\geq 2F''_\circ \sum_i ((\boldsymbol{\theta}_* - \boldsymbol{\theta}'_*)^\top \boldsymbol{x}_i)^2 \qquad (59) \\ &= 2F''_\circ (\boldsymbol{\theta}_* - \boldsymbol{\theta}'_*)^\top \mathrm{S}\mathrm{S}^\top (\boldsymbol{\theta}_* - \boldsymbol{\theta}'_*) \ , \qquad (60) \end{aligned}$$

where matrix $\mathrm{S} \in \mathbb{R}^{d \times m}$ is formed by the observations of $\mathcal{S}_{|y}$ in columns, and ineq. (59) comes from (51). Define $\mathrm{T} \doteq (d / \sum_i \|\boldsymbol{x}_i\|_2^2) \mathrm{S}\mathrm{S}^\top$. Its trace satisfies $\mathrm{tr}\,(\mathrm{T}) = d$. Let $\lambda_d \geq \lambda_{d-1} \geq ... \geq \lambda_1 > 0$

denote eigenvalues of T, with λ_1 strictly positive because $\mathrm{S}\mathrm{S}^\top = \mathrm{V}^\top\mathrm{V} \succ 0$. The AGH inequality brings:

$$\begin{aligned}\prod_2^d \lambda_k &\leq \left(\frac{1}{d-1}\sum_{k=2}^d \lambda_k\right)^{d-1} &(61)\\ &= \left(\frac{\mathrm{tr}\,(\mathrm{T})-\lambda_1}{d-1}\right)^{d-1}\\ &= \left(\frac{d-\lambda_1}{d-1}\right)^{d-1}\\ &\leq \left(\frac{d}{d-1}\right)^{d-1} . &(62)\end{aligned}$$

Multiplying both side by λ_1 and rearranging yields:

$$\lambda_1 \geq \left(\frac{d-1}{d}\right)^{d-1} \det \mathrm{T} \quad (63)$$

Let $\lambda_\circ > 0$ denote the minimal eigenvalue of $\mathrm{S}\mathrm{S}^\top$. It satisfies $\lambda_\circ = (\sum_i \|\boldsymbol{x}_i\|_2^2/d)\lambda_1$ and thus it comes from ineq. (63):

$$\begin{aligned}\lambda_\circ &\geq \left(\frac{d-1}{d}\right)^{d-1}\left(\frac{d}{\sum_i \|\boldsymbol{x}_i\|_2^2}\right)^{d-1} \det \mathrm{S}\mathrm{S}^\top\\ &= \left(\frac{d-1}{d}\right)^{d-1} \det\left[\left(\frac{d}{\sum_i \|\boldsymbol{x}_i\|_2^2}\right)^{1-\frac{1}{d}} \mathrm{S}\mathrm{S}^\top\right]\\ &= \left(\frac{d-1}{d}\right)^{d-1} \det \tilde{\mathrm{V}}^\top\tilde{\mathrm{V}} &(64)\\ &= \left(\frac{d-1}{d}\right)^{d-1} \mathrm{vol}^2(\tilde{\mathrm{V}}) &(65)\\ &\geq \frac{1}{e}\mathrm{vol}^2(\tilde{\mathrm{V}}) . &(66)\end{aligned}$$

We have used notation $\mathrm{vol}(\tilde{\mathrm{V}}) \doteq \sqrt{\det \tilde{\mathrm{V}}^\top\tilde{\mathrm{V}}}$. Since $(\boldsymbol{\theta}_* - \boldsymbol{\theta}'_*)^\top \mathrm{S}\mathrm{S}^\top(\boldsymbol{\theta}_* - \boldsymbol{\theta}'_*) \geq \lambda_\circ\|\boldsymbol{\theta}_* - \boldsymbol{\theta}'_*\|_2^2$, combining (60) with (66) yields the following lowerbound on a:

$$a \geq \frac{2}{e}F''_\circ \mathrm{vol}^2(\tilde{\mathrm{V}})\|\boldsymbol{\theta}_* - \boldsymbol{\theta}'_*\|_2^2 . \quad (67)$$

Going back to (56), we get

$$\lambda\|\boldsymbol{\theta}_* - \boldsymbol{\theta}'_*\|_2^2 - \frac{1}{2}\left(\boldsymbol{\mu} - \boldsymbol{\mu}'\right)^\top(\boldsymbol{\theta}_* - \boldsymbol{\theta}'_*) + \frac{b}{em}F''_\circ \mathrm{vol}^2(\tilde{\mathrm{V}})\|\boldsymbol{\theta}_* - \boldsymbol{\theta}'_*\|_2^2 \leq 0 .$$

Since $\left(\boldsymbol{\mu} - \boldsymbol{\mu}'\right)^\top(\boldsymbol{\theta}_* - \boldsymbol{\theta}'_*) \leq \|\boldsymbol{\mu} - \boldsymbol{\mu}'\|_2\|\boldsymbol{\theta}_* - \boldsymbol{\theta}'_*\|_2$, we get after chaining the inequalities and solving for $\|\boldsymbol{\theta}_* - \boldsymbol{\theta}'_*\|_2$:

$$\|\boldsymbol{\theta}_* - \boldsymbol{\theta}'_*\|_2 \leq \frac{1}{2\lambda + \frac{2}{em}F''_\circ \mathrm{vol}^2(\tilde{\mathrm{V}})}\|\boldsymbol{\mu} - \boldsymbol{\mu}'\|_2 ,$$

as claimed. ■

The second Lemma is used to (51) when $F(x) = F_\phi$. Notice that we cannot rely on strong convexity arguments on F_ϕ, as this do not hold in general. The Lemma is stated in a more general setting than for just $F = F_\phi$.

Lemma 5 *Fix $\lambda, b > 0$, and let $x_* \doteq \max_i \|\boldsymbol{x}_i\|_2$. Suppose that $\|\boldsymbol{\mu}\|_2 \leq \mu_*$ for some $\mu > 0$. Let*

$$F(\mathcal{S}_{|y}, \boldsymbol{\theta}, \boldsymbol{\mu}, \lambda) \quad = \quad \frac{b}{2m}\left(\sum_i \sum_\sigma F(\sigma \boldsymbol{\theta}^\top \boldsymbol{x}_i)\right) - \frac{1}{2}\boldsymbol{\theta}^\top \boldsymbol{\mu} + \lambda \|\boldsymbol{\theta}\|_2^2 \ , \tag{68}$$

and let $\boldsymbol{\theta}_* \doteq \arg\min_{\boldsymbol{\theta}} F(\mathcal{S}_{|y}, \boldsymbol{\theta}, \boldsymbol{\mu}, \lambda)$. Suppose that $F(.)$ is L-Lipschitz. Then

$$\|\boldsymbol{\theta}_*\|_2 \quad \leq \quad \frac{bLx_* + \mu_*}{\lambda} \ . \tag{69}$$

Proof Let us define a shrinking of the optimal solution $\boldsymbol{\theta}_*$, $\boldsymbol{\theta}_\alpha \doteq \alpha \boldsymbol{\theta}_*$ for $\alpha \in (0,1)$. We have

$$\begin{aligned}
F(\mathcal{S}_{|y}, \boldsymbol{\theta}_\alpha, \boldsymbol{\mu}, \lambda) &= \frac{b}{2m}\left(\sum_i \sum_\sigma F(\sigma \boldsymbol{\theta}_\alpha^\top \boldsymbol{x}_i)\right) - \frac{1}{2}\boldsymbol{\theta}_\alpha^\top \boldsymbol{\mu} + \lambda \|\boldsymbol{\theta}_\alpha\|_2^2 \\
&= \frac{b}{2m}\left(\sum_i \sum_\sigma F(\sigma\alpha \boldsymbol{\theta}_*^\top \boldsymbol{x}_i)\right) - \frac{\alpha}{2}\boldsymbol{\theta}_*^\top \boldsymbol{\mu} + \lambda\alpha^2 \|\boldsymbol{\theta}_*\|_2^2 \\
&\leq \frac{b}{2m}\left(\sum_i \sum_\sigma F(\sigma \boldsymbol{\theta}_*^\top \boldsymbol{x}_i) + L\left|\sigma\alpha \boldsymbol{\theta}_*^\top \boldsymbol{x}_i - \sigma \boldsymbol{\theta}_*^\top \boldsymbol{x}_i\right|\right) + -\frac{\alpha}{2}\boldsymbol{\theta}_*^\top \boldsymbol{\mu} \\
&\quad + \lambda\alpha^2 \|\boldsymbol{\theta}_*\|_2^2 \qquad (70) \\
&= \frac{b}{2m}\left(\sum_i \sum_\sigma F(\sigma \boldsymbol{\theta}_*^\top \boldsymbol{x}_i)\right) + \frac{bK(1-\alpha)}{m}\sum_i |\boldsymbol{\theta}_*^\top \boldsymbol{x}_i| - \frac{\alpha}{2}\boldsymbol{\theta}_*^\top \boldsymbol{\mu} \\
&\quad + \lambda\alpha^2 \|\boldsymbol{\theta}_*\|_2^2 \ , \qquad (71)
\end{aligned}$$

where (70) holds because F is L-Lipschitz. To have eq. (71) smaller than $F(\mathcal{S}_{|y}, \boldsymbol{\theta}_*, \boldsymbol{\mu}, \lambda)$, we need equivalently:

$$\frac{bL(1-\alpha)}{m}\sum_i |\boldsymbol{\theta}_*^\top \boldsymbol{x}_i| - \frac{\alpha}{2}\boldsymbol{\theta}_*^\top \boldsymbol{\mu} + \lambda\alpha^2 \|\boldsymbol{\theta}_*\|_2^2 \quad \leq \quad -\frac{1}{2}\boldsymbol{\theta}_*^\top \boldsymbol{\mu} + \lambda \|\boldsymbol{\theta}_*\|_2^2 \ ,$$

that is:

$$\frac{bL(1-\alpha)}{m}\sum_i |\boldsymbol{\theta}_*^\top \boldsymbol{x}_i| + \frac{1-\alpha}{2}\boldsymbol{\theta}_*^\top \boldsymbol{\mu} \quad \leq \quad \lambda(1-\alpha^2)\|\boldsymbol{\theta}_*\|_2^2 \ ,$$

and to find an $\alpha \in (0,1)$ such that this holds, because of Cauchy-Schwartz inequality, it is sufficient that $(1-\alpha)(bLx_* + \mu) \leq \lambda(1-\alpha^2)\|\boldsymbol{\theta}_*\|_2$, *i.e.*:

$$\|\boldsymbol{\theta}_*\|_2 \quad \geq \quad \frac{bLx_* + \|\boldsymbol{\mu}\|_2}{\lambda(1+\alpha)} \ .$$

Hence, whenever $\|\boldsymbol{\theta}_*\|_2 > (bLx_* + \|\boldsymbol{\mu}\|_2)/\lambda$, there is a shrinking of the optimal solution to eq. (68) that further decreases the risk, thus contradicting its optimality. This ends the proof of Lemma 5. ■

Notice that Lemma 5 does not require $F(x)$ to be convex, nor differentiable. To use this Lemma, remark that for any F_ϕ,

$$F'_\phi(x) \quad = \quad -\frac{1}{b_\phi}(\phi^\star)'(-x) = -\frac{1}{b_\phi}(\phi')^{-1}(-x) \in [-1/b_\phi, 0] \ , \tag{72}$$

for any $x \in \phi'([0,1])$ [2], and thus F_ϕ is $1/b_\phi$-Lipschitz. Finally, considering (51), for any $\alpha \in [0,1]$

$$\begin{aligned}
|\pm(\alpha\boldsymbol{\theta}_* + (1-\alpha)\boldsymbol{\theta}'_*)^\top \boldsymbol{x}_i| &\leq (\alpha\|\boldsymbol{\theta}_*\|_2 + (1-\alpha)\|\boldsymbol{\theta}'_*\|_2)x_* \\
&\leq \frac{x_* + \alpha\|\boldsymbol{\mu}\|_2 + (1-\alpha)\|\boldsymbol{\mu}'\|_2}{\lambda} \qquad (73) \\
&\leq \frac{x_* + \max\{\|\boldsymbol{\mu}\|_2, \|\boldsymbol{\mu}'\|_2\}}{\lambda} \ , \qquad (74)
\end{aligned}$$

where ineq. (73) uses Lemma 5 with $b = 1/K = b_\phi$. $\boldsymbol{\mu}$ and $\boldsymbol{\mu}'$ are the parameters of $F(\mathcal{S}_{|y}, ., \boldsymbol{\mu}, \lambda)$ and $F(\mathcal{S}_{|y}, ., \boldsymbol{\mu}', \lambda)$ in Lemma 4.

Algorithm 1 Label Assignation (LA)

Input $\boldsymbol{\theta} \in \mathbb{R}^d$, a bag $\mathcal{B} = \{\boldsymbol{x}_i \in \mathbb{R}^d, i = 1, 2, ..., m\}$, bag size $m^+ \in [m]$;
If $\mathcal{B} = \emptyset$ **then** stop
Else if $m^+ \notin (m)$ **then** $y_i \leftarrow \mathrm{I}(m^+ = m) - \mathrm{I}(m^+ = 0), \forall i = 1, 2, ..., m$
Else
Step 1 : $i^* \leftarrow \arg\max_i |\boldsymbol{\theta}^\top \boldsymbol{x}_i|$
Step 2 : $y_{i^*} \leftarrow \mathrm{sign}(\boldsymbol{\theta}^\top \boldsymbol{x}_{i^*})$
Step 3 : LA$(\boldsymbol{\theta}, \mathcal{B}\backslash\{\boldsymbol{x}_{i^*}\}, m^+ - \mathrm{I}(y_{i^*} = 1))$

Now, going back to the parameters of Theorem 6, we make the change $\boldsymbol{\mu} \to \boldsymbol{\mu}_{\mathbb{S}}$ and $\boldsymbol{\mu}' \to \tilde{\boldsymbol{\mu}}_{\mathbb{S}}$ and obtain the statement of the Theorem for interval

$$\mathbb{I} \quad = \quad [\pm(x_* + \max\{\|\boldsymbol{\mu}_{\mathbb{S}}\|_2, \|\tilde{\boldsymbol{\mu}}_{\mathbb{S}}\|_2\})] \quad . \tag{75}$$

This achieves the proof of Theorem 6.

2.8 Proof of Lemma 7

We make the proof for optimization strategy OPT $= \min$. The case OPT $= \max$ flips the choice of the label in Step 2. To minimize $F_\phi(\mathbb{S}_{|y}, \boldsymbol{\theta}_t, \boldsymbol{\mu}_{\mathbb{S}}(\boldsymbol{\sigma}))$ over $\boldsymbol{\sigma} \in \Sigma_{\tilde{\boldsymbol{\pi}}}$, we just have to find $\boldsymbol{\sigma}_* \in \arg\max_{\boldsymbol{\sigma} \in \Sigma_{\tilde{\boldsymbol{\pi}}}} \boldsymbol{\theta}^\top \sum_i \sigma_i \boldsymbol{x}_i$, and we can do that bag-wise. Algorithm 1 presents the labeling (notation $(m) \doteq \{1, 2, ..., m-1\}$). Remark that the time complexity for one bag is $O(m_j \log m_j)$ due to the ordering (Step 1), so the overall complexity is indeed $O(m \max_i \log m_i)$.

Lemma 6 *Let $\boldsymbol{\sigma}_* \doteq \{\sigma_1^*, \sigma_2^*, ..., \sigma_m^*\}$ be the set of labels obtained after running* LA$(\boldsymbol{\theta}, \mathbb{S}_j, m_j^+)$ *for $j = 1, 2, ..., n$. Then $\boldsymbol{\sigma}_* \in \arg\max_{\boldsymbol{\sigma} \in \Sigma_{\tilde{\boldsymbol{\pi}}}} \boldsymbol{\theta}^\top \sum_i \sigma_i \boldsymbol{x}_i$.*

Proof The total edge, $\boldsymbol{\theta}^\top \sum_i \sigma_i \boldsymbol{x}_i$ (for any $\boldsymbol{\sigma} \in \Sigma_{\tilde{\boldsymbol{\pi}}}$), can be summable bag-wise wrt the coordinates of $\boldsymbol{\sigma}$. Consider thus the optimal set $\{\boldsymbol{\sigma}^\star\}_{\mathcal{B}} \doteq \arg\max_{\boldsymbol{\sigma} \in \{-1,1\}^{m'} : \mathbf{1}^\top \boldsymbol{\sigma} = 2m^+ - m'} \boldsymbol{\theta}^\top \sum_{\boldsymbol{x}_i \in \mathcal{B}} \sigma_i \boldsymbol{x}_i$, for some bag $\mathcal{B} = \{\boldsymbol{x}_i, i = 1, 2, ..., m'\}$, with constraint $m^+ \in [m']$. This set contains the label assignment $\boldsymbol{\sigma}_*$ returned by LA$(\boldsymbol{\theta}, \mathcal{B}, m^+)$, a property that follows from two simple observations:

- **P1** Consider any observation $\boldsymbol{x}_i$ of bag $\mathcal{B}$; for any optimal labeling $\boldsymbol{\sigma}^\star$ of $\mathcal{B}$, let $m'^+ \doteq m^+ - \mathrm{I}(\sigma_i^\star = 1)$. Define the set $\{\boldsymbol{\sigma}'^\star\}_i$ of optimal labelings of $\mathcal{B}\backslash\{\boldsymbol{x}_i\}$ with constraint $m'^+ \doteq m^+ - \mathrm{I}(\sigma_i^\star = 1)$. Then this set coincides with the set created by taking the elements of $\{\boldsymbol{\sigma}^\star\}_{\mathcal{B}}$ to which we drop coordinate i. This follows from the per-observation summability of the total edge wrt labels.
- **P2** Assume $m^+ \in (m')$. $\forall i^* \in \arg\max_i |\boldsymbol{\theta}^\top \boldsymbol{x}_i|$, there exists an optimal assignment $\boldsymbol{\sigma}^\star$ such that $\sigma_{i^*}^\star = \mathrm{sign}(\boldsymbol{\theta}^\top \boldsymbol{x}_{i^*})$. Otherwise, starting from any optimal assignment $\boldsymbol{\sigma}^\star$, we can flip the label of $\boldsymbol{x}_{i^*}$ and the label of any other $\boldsymbol{x}_i$ for which $\sigma_i^\star \neq \sigma_{i^*}^\star$, and get a label assignment that satisfies constraint m^+ and cannot be worse than $\boldsymbol{\sigma}^\star$, and is thus optimal, a contradiction.

Hence, LA$(\boldsymbol{\theta}, \mathcal{B}, m^+)$ picks at each iteration a label that matches one in a subset of optimal labelings, and the recursive call preserves the subset of optimal labelings. Since when $m^+ \notin (m)$ the solution returned by LA$(\boldsymbol{\theta}, \mathcal{B}, m^+)$ is obviously optimal, we end up when the current $\mathcal{B}$ is empty with $\boldsymbol{\sigma}_* \in \arg\max_{\boldsymbol{\sigma} \in \Sigma_{\tilde{\boldsymbol{\pi}}}} \boldsymbol{\theta}^\top \sum_i \sigma_i \boldsymbol{x}_i$, as claimed. ■

2.9 Proof of Theorem 8

We prove separately Eqs (14) and (15).

2.9.1 Proof of eq. (14)

Notations : unless explicitly stated, all samples like $\mathcal{S}$ and $\mathcal{S}'$ are of size m. To make the reading of our expectations clear and simple, we shall write $\mathbb{E}_{\mathcal{D}}$ for $\mathbb{E}_{(\boldsymbol{x},y)\sim\mathcal{D}}$, $\mathbb{E}_{\Sigma_m}$ for $\mathbb{E}_{\boldsymbol{\sigma}\sim\Sigma_m}$, $\mathbb{E}_{\mathcal{S}}$ for $\mathbb{E}_{(\boldsymbol{x},y)\sim\mathcal{S}}$, $\mathbb{E}_{\mathcal{D}'_m}$ for $\mathbb{E}_{\mathcal{S}'\sim\mathcal{D}}$ and $\mathbb{E}_{\mathcal{D}_m}$ for $\mathbb{E}_{\mathcal{S}\sim\mathcal{D}}$.

We now proceed to the proof, that follows the same main steps as that of Theorem 5 in [6]. For any $q \in [0,1]$, let us define the convex combination:

$$F_\phi(q, h(\boldsymbol{x})) \doteq qF_\phi(h(\boldsymbol{x})) + (1-q)F_\phi(-h(\boldsymbol{x})) \ . \tag{76}$$

It follows that

$$\mathbb{E}_{\Sigma_{\hat{\boldsymbol{\pi}}}}\mathbb{E}_{\mathcal{S}}[F_\phi(\sigma(\boldsymbol{x})h(\boldsymbol{x}))] = \mathbb{E}_{\mathcal{S}}[F_\phi(\hat{\pi}(\boldsymbol{x}), h(\boldsymbol{x}))] \ , \tag{77}$$

with $\hat{\pi}(\boldsymbol{x})$ the label proportion of the bag to which $\boldsymbol{x}$ belongs in $\mathcal{S}$. We also have $\forall h$,

$$\mathbb{E}_{\mathcal{D}}[F_\phi(yh(\boldsymbol{x}))] \leq \mathbb{E}_{\mathcal{S}}[F_\phi(\hat{\pi}(\boldsymbol{x}), h(\boldsymbol{x}))] + \Lambda(\mathcal{S}) \ , \tag{78}$$

with

$$\Lambda(\mathcal{S}) \doteq \sup_g \{\mathbb{E}_{\mathcal{D}}[F_\phi(yg(\boldsymbol{x}))] - \mathbb{E}_{\mathcal{S}}[F_\phi(\hat{\pi}(\boldsymbol{x}), g(\boldsymbol{x}))]\} \ . \tag{79}$$

Let us bound the deviations of $\Lambda(\mathcal{S})$ around its expectation on the sampling of $\mathcal{S}$, using the independent bounded differences inequality (IBDI, [7]). for which we need to upperbound the maximum difference for the supremum term computed over two samples $\mathcal{S}$ and $\mathcal{S}'$ of the same size, such that $\mathcal{S}'$ is $\mathcal{S}$ with one example replaced. We have:

$$|\Lambda(\mathcal{S}) - \Lambda(\mathcal{S}')| \leq |\mathbb{E}_{\mathcal{S}}[F_\phi(\hat{\pi}(\boldsymbol{x}), g(\boldsymbol{x}))] - \mathbb{E}_{\mathcal{S}'}[F_\phi(\hat{\pi}'(\boldsymbol{x}), g(\boldsymbol{x}))]| \ , \tag{80}$$

with $\hat{\boldsymbol{\pi}}$ and $\hat{\boldsymbol{\pi}}'$ denoting the corresponding label proportions in $\mathcal{S}$ and $\mathcal{S}'$. Let $\{\boldsymbol{x}_1\} = \mathcal{S}\backslash\mathcal{S}'$ and $\{\boldsymbol{x}_2\} = \mathcal{S}'\backslash\mathcal{S}$. Let $\boldsymbol{x}_1 \in \mathcal{S}_j$ and $\boldsymbol{x}_2 \in \mathcal{S}'_{j'}$ for some bags j and j'. Upperbound (80) depends only on bags j and j'. For any $\boldsymbol{x} \in (\mathcal{S}_j \cup \mathcal{S}_{j'})\backslash\{\boldsymbol{x}_1, \boldsymbol{x}_2\}$, eqs. (2) and (3) bring:

$$\begin{aligned} F_\phi(\hat{\pi}(\boldsymbol{x}), g(\boldsymbol{x})) - F_\phi(\hat{\pi}'(\boldsymbol{x}), g(\boldsymbol{x})) &\leq \frac{|F_\phi(g(\boldsymbol{x})) - F_\phi(-g(\boldsymbol{x}))|}{m(\boldsymbol{x})} \\ &= \frac{|g(\boldsymbol{x})|}{b_\phi m(\boldsymbol{x})} \qquad (81) \\ &\leq \frac{h_*}{b_\phi m(\boldsymbol{x})} \ , \qquad (82) \end{aligned}$$

where $m(\boldsymbol{x})$ is the size of the bag to which it belongs in $\mathcal{S}$, plus 1 iff it is bag j' and $j' \neq j$, minus 1 iff it is bag j and $j' \neq j$. Furthermore, (2) and (3) also bring:

$$\begin{aligned} F_\phi(\hat{\pi}(\boldsymbol{x}), g(\boldsymbol{x})) &= F_\phi(|g(\boldsymbol{x})|) + \frac{1}{b_\phi}((1-\hat{\pi}(\boldsymbol{x}))1_{g(\boldsymbol{x})>0} + \hat{\pi}(\boldsymbol{x})(1-1_{g(\boldsymbol{x})>0}))|g(\boldsymbol{x})| \\ &\leq F_\phi(0) + \frac{1}{b_\phi}((1-\hat{\pi}(\boldsymbol{x}))1_{g(\boldsymbol{x})>0} + \hat{\pi}(\boldsymbol{x})(1-1_{g(\boldsymbol{x})>0}))h^* \\ &\leq F_\phi(0) + \frac{h^*}{b_\phi} \ , \forall \boldsymbol{x} \in \mathcal{S} \ . \end{aligned}$$

Also, it comes from its definition that:

$$\begin{aligned} F_\phi(0) &= \frac{1}{b_\phi}(0\phi'^{-1}(0) - \phi(\phi'^{-1}(0))) \\ &= \frac{-\phi(1/2)}{b_\phi} = 1 \ . \qquad (83) \end{aligned}$$

We obtain that:

$$\begin{aligned} |\Lambda(\mathcal{S}) - \Lambda(\mathcal{S}')| &\leq \frac{1}{m}\left(1 + \frac{h^*}{b_\phi} + 1 + \frac{h^*}{b_\phi}\right) + \frac{1}{m}\sum_{\boldsymbol{x}\in(\mathcal{S}_j\cup\mathcal{S}_{j'})\backslash\{\boldsymbol{x}_1,\boldsymbol{x}_2\}} \frac{h_*}{b_\phi m(\boldsymbol{x})} \\ &\leq \frac{Q_1}{m} \ , \qquad (84) \end{aligned}$$

where

$$Q_1 \doteq 2\left(\frac{2h_*}{b_\phi}+1\right) . \tag{85}$$

So the IBDI yields that with probability $\leq \delta/2$ over the sampling of $\mathcal{S}$,

$$\Lambda(\mathcal{S}) \geq \mathbb{E}_{\mathcal{D}_m} \sup_g \{\mathbb{E}_{\mathcal{D}}[F_\phi(yg(\boldsymbol{x}))] - \mathbb{E}_{\mathcal{S}}[F_\phi(\hat{\pi}(\boldsymbol{x}), g(\boldsymbol{x}))]\} + Q_1\sqrt{\frac{1}{2m}\log\frac{2}{\delta}} , \tag{86}$$

We now upperbound the expectation in (86). Using the convexity of the supremum, we have

$$\begin{aligned}
&\mathbb{E}_{\mathcal{D}_m} \sup_g \{\mathbb{E}_{\mathcal{D}}[F_\phi(yg(\boldsymbol{x}))] - \mathbb{E}_{\mathcal{S}}[F_\phi(\hat{\pi}(\boldsymbol{x}), g(\boldsymbol{x}))]\} \\
&= \mathbb{E}_{\mathcal{D}_m} \sup_g \left\{\mathbb{E}_{\mathcal{D}'_m}[F_\phi(yg(\boldsymbol{x}))] - \mathbb{E}_{\mathcal{S}}[F_\phi(\hat{\pi}(\boldsymbol{x}), g(\boldsymbol{x}))]\right\} \\
&\leq \mathbb{E}_{\mathcal{D}_m, \mathcal{D}'_m} \sup_g \{\mathbb{E}_{\mathcal{S}'}[F_\phi(yg(\boldsymbol{x}))] - \mathbb{E}_{\mathcal{S}}[F_\phi(\hat{\pi}(\boldsymbol{x}), g(\boldsymbol{x}))]\} .
\end{aligned} \tag{87}$$

Consider any set $\mathcal{S} \sim \mathcal{D}_{2m}$, and let $\mathcal{I}' \subseteq [2m]$ be a subset of m indices, picked uniformly at random among all $\binom{2m}{m}$ possible choices. For any $\mathcal{I} \subseteq [2m]$, let $\mathcal{S}(\mathcal{I})$ denote the subset of examples whose index matches $\mathcal{I}$, and for any $\boldsymbol{x} \in \mathcal{S}(\mathcal{I})$, let $\hat{\pi}(\boldsymbol{x}|\mathcal{S}(\mathcal{I}))$ denote its bag proportion in $\mathcal{S}(\mathcal{I})$. For any $\mathcal{I}'_l$ indexed by $l \geq 1$ and any $\boldsymbol{x} \in \mathcal{S}$, let:

$$\hat{\pi}^s_{|l}(\boldsymbol{x}) \doteq \begin{cases} \hat{\pi}(\boldsymbol{x}|\mathcal{S}(\mathcal{I}'_l)) & \text{if} \quad \boldsymbol{x} \in \mathcal{S}(\mathcal{I}'_l) \\ \hat{\pi}(\boldsymbol{x}|\mathcal{S}\backslash\mathcal{S}(\mathcal{I}'_l)) & \text{otherwise} \end{cases} \tag{88}$$

denote the label proportions induced by the split of $\mathcal{S}$ in two subsamples $\mathcal{S}(\mathcal{I}'_l)$ and $\mathcal{S}\backslash\mathcal{S}(\mathcal{I}'_l)$. Let

$$\hat{\pi}^\ell_{|l}(\boldsymbol{x}) \doteq \begin{cases} y & \text{if} \quad \boldsymbol{x} \in \mathcal{S}(\mathcal{I}'_l) \\ \hat{\pi}(\boldsymbol{x}|\mathcal{S}\backslash\mathcal{S}(\mathcal{I}'_l)) & \text{otherwise} \end{cases} , \tag{89}$$

where y is the true label of $\boldsymbol{x}$. Let $\sigma_l(\boldsymbol{x}) \doteq 2 \times 1_{\boldsymbol{x} \in \mathcal{S}(\mathcal{I}'_l)} - 1$. The Label Proportion Complexity (LPC) L_{2m} quantifies the discrepance between these two estimators. When each bag in $\mathcal{S}$ has label proportion zero or one, each term factoring classifier h in eq. (13) (main file) is zero, so $L_{2m} = 0$.

Lemma 7 *The following holds true:*

$$\begin{aligned}
&\mathbb{E}_{\mathcal{D}_m, \mathcal{D}'_m} \sup_g \{\mathbb{E}_{\mathcal{S}'}[F_\phi(yg(\boldsymbol{x}))] - \mathbb{E}_{\mathcal{S}}[F_\phi(\hat{\pi}(\boldsymbol{x}), g(\boldsymbol{x}))]\} \\
&\leq 2\mathbb{E}_{\mathcal{D}_m, \Sigma_m} \sup_h \{\mathbb{E}_{\mathcal{S}}[\sigma(\boldsymbol{x})F_\phi(\hat{\pi}(\boldsymbol{x}), h(\boldsymbol{x}))]\} + L_{2m} .
\end{aligned} \tag{90}$$

Proof For any $\boldsymbol{\sigma} \in \Sigma_m$ and any sets $\mathcal{S} = \{\boldsymbol{x}_1, \boldsymbol{x}_2, ..., \boldsymbol{x}_m\}$ and $\mathcal{S}' = \{\boldsymbol{x}'_1, \boldsymbol{x}'_2, ..., \boldsymbol{x}'_m\}$of size m, denote

$$\begin{aligned}
\mathcal{S}_{\boldsymbol{\sigma}} &\doteq \{\boldsymbol{x}'_i \text{ iff } \sigma_i = 1, \boldsymbol{x}_i \text{ otherwise}\} , \\
\mathcal{S}_{\overline{\boldsymbol{\sigma}}} &\doteq \{\boldsymbol{x}'_i \text{ iff } \sigma_i = -1, \boldsymbol{x}_i \text{ otherwise}\} = (\mathcal{S} \cup \mathcal{S}')\backslash\mathcal{S}_{\boldsymbol{\sigma}} .
\end{aligned} \tag{91}$$

and

$$\hat{\pi}_*(\boldsymbol{x}) \doteq \begin{cases} \hat{\pi}_{\boldsymbol{\sigma}}(\boldsymbol{x}) & \text{if} \quad \boldsymbol{x} \in \mathcal{S}_{\boldsymbol{\sigma}} , \\ \hat{\pi}_{\overline{\boldsymbol{\sigma}}}(\boldsymbol{x}) & \text{otherwise} \end{cases} , \tag{92}$$

where $\hat{\pi}_{\boldsymbol{\sigma}}(.)$ denote the label proportions in $\mathcal{S}_{\boldsymbol{\sigma}}$ and $\hat{\pi}_{\overline{\boldsymbol{\sigma}}}(.)$ denote the label proportions in $\mathcal{S}_{\overline{\boldsymbol{\sigma}}}$. Let $\hat{\pi}(.)$ denote the label proportions in $\mathcal{S}$, $\hat{\pi}'(.)$ denote the label proportions in $\mathcal{S}'$ (we know each bag to which each example in $\mathcal{S}'$ belongs to, so we can compute these estimators), We have

$$\begin{aligned}
&\mathbb{E}_{\mathcal{D}_m, \mathcal{D}'_m} \sup_h \{\mathbb{E}_{\mathcal{S}'}[F_\phi(yh(\boldsymbol{x}))] - \mathbb{E}_{\mathcal{S}}[F_\phi(\hat{\pi}(\boldsymbol{x}), h(\boldsymbol{x}))]\} \\
&= \mathbb{E}_{\mathcal{D}_m, \mathcal{D}'_m} \sup_h \left\{\mathbb{E}_{\mathcal{S}'}[F_\phi(\hat{\pi}'(\boldsymbol{x}), h(\boldsymbol{x}))] - \mathbb{E}_{\mathcal{S}}[F_\phi(\hat{\pi}(\boldsymbol{x}), h(\boldsymbol{x}))] - \frac{1}{b_\phi} \times \Delta_1\right\} \\
&= \mathbb{E}_{\mathcal{D}_m, \mathcal{D}'_m} \sup_h \left\{\mathbb{E}_{\mathcal{S}_{\boldsymbol{\sigma}}}[\sigma(\boldsymbol{x})F_\phi(\hat{\pi}^l(\boldsymbol{x}), h(\boldsymbol{x}))] - \mathbb{E}_{\mathcal{S}_{\overline{\boldsymbol{\sigma}}}}[\sigma(\boldsymbol{x})F_\phi(\hat{\pi}^r(\boldsymbol{x}), h(\boldsymbol{x}))] - \frac{1}{b_\phi} \times \Delta_1\right\}
\end{aligned} \tag{93}$$

with

$$\Delta_1 \doteq \mathbb{E}_{\mathcal{S}'}[((1-\hat{\pi}'(\boldsymbol{x}))1_{y=1} - \hat{\pi}'(\boldsymbol{x})1_{y=-1})h(\boldsymbol{x})] \ ; \tag{94}$$

$$\hat{\pi}^l(\boldsymbol{x}) \doteq \frac{1}{2}\left((1+\sigma(\boldsymbol{x}))\hat{\pi}'(\boldsymbol{x}) + (1-\sigma(\boldsymbol{x}))\hat{\pi}(\boldsymbol{x})\right) \ ,$$

$$\hat{\pi}^r(\boldsymbol{x}) \doteq \frac{1}{2}\left((1+\sigma(\boldsymbol{x}))\hat{\pi}(\boldsymbol{x}) + (1-\sigma(\boldsymbol{x}))\hat{\pi}'(\boldsymbol{x})\right) \ . \tag{95}$$

We also have from eq. (2) and (3):

$$\mathbb{E}_{\mathcal{S}_{\boldsymbol{\sigma}}}[\sigma(\boldsymbol{x})F_\phi(\hat{\pi}^l(\boldsymbol{x}), h(\boldsymbol{x}))] = \mathbb{E}_{\mathcal{S}_{\boldsymbol{\sigma}}}[\sigma(\boldsymbol{x})F_\phi(\hat{\pi}_{\boldsymbol{\sigma}}(\boldsymbol{x}), h(\boldsymbol{x}))] - \frac{1}{b_\phi} \times \Delta_2 \ , \tag{96}$$

$$\mathbb{E}_{\mathcal{S}_{\overline{\boldsymbol{\sigma}}}}[\sigma(\boldsymbol{x})F_\phi(\hat{\pi}^r(\boldsymbol{x}), h(\boldsymbol{x}))] = \mathbb{E}_{\mathcal{S}_{\overline{\boldsymbol{\sigma}}}}[\sigma(\boldsymbol{x})F_\phi(\hat{\pi}_{\overline{\boldsymbol{\sigma}}}(\boldsymbol{x}), h(\boldsymbol{x}))] - \frac{1}{b_\phi} \times \Delta_3 \ , \tag{97}$$

with

$$\Delta_2 \doteq \mathbb{E}_{\mathcal{S}_{\boldsymbol{\sigma}}}[\sigma(\boldsymbol{x})(\hat{\pi}^l(\boldsymbol{x}) - \hat{\pi}_{\boldsymbol{\sigma}}(\boldsymbol{x}))h(\boldsymbol{x})] \ , \tag{98}$$

$$\Delta_3 \doteq \mathbb{E}_{\mathcal{S}_{\overline{\boldsymbol{\sigma}}}}[\sigma(\boldsymbol{x})(\hat{\pi}^r(\boldsymbol{x}) - \hat{\pi}_{\overline{\boldsymbol{\sigma}}}(\boldsymbol{x}))h(\boldsymbol{x})] \ . \tag{99}$$

We also have:

$$\begin{aligned}\Delta_3 - \Delta_2 - \Delta_1 &= \mathbb{E}_{\mathcal{S}'}[(\hat{\pi}_*(\boldsymbol{x}) - 1_{y=1})h(\boldsymbol{x})] + \mathbb{E}_{\mathcal{S}}[(\hat{\pi}(\boldsymbol{x}) - \hat{\pi}_*(\boldsymbol{x}))h(\boldsymbol{x})] \\ &\doteq \Delta_4 \ .\end{aligned} \tag{100}$$

Putting eqs (93), (96), (97) and (100) altogether, we get, after introducing Rademacher variables:

$$\begin{aligned}&\mathbb{E}_{\mathcal{D}_m, \mathcal{D}'_m, \Sigma_m} \sup_h \{\mathbb{E}_{\mathcal{S}'}[F_\phi(yh(\boldsymbol{x}))] - \mathbb{E}_{\mathcal{S}}[F_\phi(\hat{\pi}(\boldsymbol{x}), h(\boldsymbol{x}))]\} \\ &= \mathbb{E}_{\mathcal{D}_m, \mathcal{D}'_m, \Sigma_m} \sup_h \{\mathbb{E}_{\mathcal{S}_{\boldsymbol{\sigma}}}[\sigma(\boldsymbol{x})F_\phi(\hat{\pi}_{\boldsymbol{\sigma}}(\boldsymbol{x}), h(\boldsymbol{x}))] - \mathbb{E}_{\mathcal{S}_{\overline{\boldsymbol{\sigma}}}}[\sigma(\boldsymbol{x})F_\phi(\hat{\pi}_{\overline{\boldsymbol{\sigma}}}(\boldsymbol{x}), h(\boldsymbol{x}))] + \Delta_4\} \\ &\leq \mathbb{E}_{\mathcal{D}_m, \mathcal{D}'_m, \Sigma_m} \sup_h \{\mathbb{E}_{\mathcal{S}_{\boldsymbol{\sigma}}}[\sigma(\boldsymbol{x})F_\phi(\hat{\pi}_{\boldsymbol{\sigma}}(\boldsymbol{x}), h(\boldsymbol{x}))] - \mathbb{E}_{\mathcal{S}_{\overline{\boldsymbol{\sigma}}}}[\sigma(\boldsymbol{x})F_\phi(\hat{\pi}_{\overline{\boldsymbol{\sigma}}}(\boldsymbol{x}), h(\boldsymbol{x}))]\} \\ &\quad + \mathbb{E}_{\mathcal{D}_m, \mathcal{D}'_m, \Sigma_m} \sup_h \{\mathbb{E}_{\mathcal{S}'}[(\hat{\pi}_*(\boldsymbol{x}) - 1_{y=1})h(\boldsymbol{x})] + \mathbb{E}_{\mathcal{S}}[(\hat{\pi}(\boldsymbol{x}) - \hat{\pi}_*(\boldsymbol{x}))h(\boldsymbol{x})]\} \\ &= \mathbb{E}_{\mathcal{D}_m, \mathcal{D}'_m, \Sigma_m} \sup_h \{\mathbb{E}_{\mathcal{S}'}[\sigma(\boldsymbol{x})F_\phi(\hat{\pi}'(\boldsymbol{x}), h(\boldsymbol{x}))] - \mathbb{E}_{\mathcal{S}}[\sigma(\boldsymbol{x})F_\phi(\hat{\pi}(\boldsymbol{x}), h(\boldsymbol{x}))]\} \\ &\quad + \mathbb{E}_{\mathcal{D}_m, \mathcal{D}'_m, \Sigma_m} \sup_h \{\mathbb{E}_{\mathcal{S}'}[(\hat{\pi}_*(\boldsymbol{x}) - 1_{y=1})h(\boldsymbol{x})] + \mathbb{E}_{\mathcal{S}}[(\hat{\pi}(\boldsymbol{x}) - \hat{\pi}_*(\boldsymbol{x}))h(\boldsymbol{x})]\} \qquad (101) \\ &\leq 2\mathbb{E}_{\mathcal{D}_m, \Sigma_m} \sup_h \{\mathbb{E}_{\mathcal{S}}[\sigma(\boldsymbol{x})F_\phi(\hat{\pi}(\boldsymbol{x}), h(\boldsymbol{x}))]\} \\ &\quad + \mathbb{E}_{\mathcal{D}_m, \mathcal{D}'_m, \Sigma_m} \sup_h \{\mathbb{E}_{\mathcal{S}'}[(\hat{\pi}_*(\boldsymbol{x}) - 1_{y=1})h(\boldsymbol{x})] + \mathbb{E}_{\mathcal{S}}[(\hat{\pi}(\boldsymbol{x}) - \hat{\pi}_*(\boldsymbol{x}))h(\boldsymbol{x})]\} \ . \qquad (102)\end{aligned}$$

Eq. (101) holds because the distribution of the supremum is the same. We also have:

$$\begin{aligned}&\mathbb{E}_{\mathcal{D}_m, \mathcal{D}'_m, \Sigma_m} \sup_h \{\mathbb{E}_{\mathcal{S}'}[(\hat{\pi}_*(\boldsymbol{x}) - 1_{y=1})h(\boldsymbol{x})] + \mathbb{E}_{\mathcal{S}}[(\hat{\pi}(\boldsymbol{x}) - \hat{\pi}_*(\boldsymbol{x}))h(\boldsymbol{x})]\} \\ &= \mathbb{E}_{\mathcal{D}_m, \mathcal{D}'_m, \Sigma_m} \sup_h \{\mathbb{E}_{\mathcal{S}}[(\hat{\pi}(\boldsymbol{x}) - \hat{\pi}_*(\boldsymbol{x}))h(\boldsymbol{x})] - \mathbb{E}_{\mathcal{S}'}[(1_{y=1} - \hat{\pi}_*(\boldsymbol{x}))h(\boldsymbol{x})]\} \\ &= \mathbb{E}_{\mathcal{D}_{2m}} \mathbb{E}_{\mathcal{J}_1^{2}, \mathcal{J}_2^{2}} \sup_h \mathbb{E}_{\mathcal{S}}[\sigma_1(\boldsymbol{x})(\hat{\pi}^s_{|2}(\boldsymbol{x}) - \hat{\pi}^\ell_{|1}(\boldsymbol{x}))h(\boldsymbol{x})] \qquad (103) \\ &= L_{2m} \ . \qquad (104)\end{aligned}$$

Eq. (103) holds because swapping the sample does not make any difference in the outer expectation, as each couple of swapped samples is generated with the same probability without swapping. Putting altogether (102) and (104) ends the proof of Lemma 7. ■

We now bound the deviations of $\mathbb{E}_{\Sigma_m} \sup_h \{\mathbb{E}_{\mathcal{S}}[\sigma(\boldsymbol{x})F_\phi(\hat{\pi}(\boldsymbol{x}), h(\boldsymbol{x}))]\}$ with respect to its expectation over the sampling of $\mathcal{S}$, $\mathbb{E}_{\mathcal{D}_m, \Sigma_m} \sup_h \{\mathbb{E}_{\mathcal{S}}[\sigma(\boldsymbol{x})F_\phi(\hat{\pi}(\boldsymbol{x}), h(\boldsymbol{x}))]\}$. To do that, we use a third time the IBDI and compute an upperbound for

$$\begin{aligned}&\left| \begin{array}{l} \mathbb{E}_{\Sigma_m} \sup_g \{\mathbb{E}_{\mathcal{S}_1}[\sigma(\boldsymbol{x})F_\phi(\hat{\pi}(\boldsymbol{x}), h(\boldsymbol{x}))]\} \\ -\mathbb{E}_{\Sigma_m} \sup_g \{\mathbb{E}_{\mathcal{S}_2}[\sigma(\boldsymbol{x})F_\phi(\hat{\pi}(\boldsymbol{x}), h(\boldsymbol{x}))]\} \end{array} \right| \\ &\leq \mathbb{E}_{\Sigma_m}\left[\left| \begin{array}{l} \sup_g \{\mathbb{E}_{\mathcal{S}_1}[\sigma(\boldsymbol{x})F_\phi(\hat{\pi}(\boldsymbol{x}), h(\boldsymbol{x}))]\} \\ -\sup_g \{\mathbb{E}_{\mathcal{S}_2}[\sigma(\boldsymbol{x})F_\phi(\hat{\pi}(\boldsymbol{x}), h(\boldsymbol{x}))]\} \end{array} \right| \right] \qquad (105) \\ &\leq \max_{\Sigma_m}\left[\left| \begin{array}{l} \sup_g \{\mathbb{E}_{\mathcal{S}_1}[\sigma(\boldsymbol{x})F_\phi(\hat{\pi}(\boldsymbol{x}), h(\boldsymbol{x}))]\} \\ -\sup_g \{\mathbb{E}_{\mathcal{S}_2}[\sigma(\boldsymbol{x})F_\phi(\hat{\pi}(\boldsymbol{x}), h(\boldsymbol{x}))]\} \end{array} \right| \right] \leq \frac{Q_1}{m} \ , \qquad (106)\end{aligned}$$

where Q_1 is defined in eq. (85). Eq. (105) holds because of the triangular inequality. Ineq. (106) holds because $|\sigma(.)| = 1$. So with probability $\leq \delta/2$ over the sampling of $\mathcal{S}$,

$$\begin{aligned}
&\mathbb{E}_{\Sigma_m} \sup_h \{\mathbb{E}_{\mathcal{S}}[\sigma(\boldsymbol{x}) F_\phi(\hat{\pi}(\boldsymbol{x}), h(\boldsymbol{x}))]\} \\
&\leq \mathbb{E}_{\mathcal{D}_m, \Sigma_m} \sup_h \{\mathbb{E}_{\mathcal{S}}[\sigma(\boldsymbol{x}) F_\phi(\hat{\pi}(\boldsymbol{x}), h(\boldsymbol{x}))]\} - Q_1 \sqrt{\frac{1}{2m} \log \frac{2}{\delta}} \ , \qquad (107)
\end{aligned}$$

where Q_1 is defined via (84). We obtain that with probability $> 1 - ((\delta/2) + (\delta/2)) = 1 - \delta$, the following holds $\forall h$:

$$\begin{aligned}
\mathbb{E}_{\mathcal{D}}[F_\phi(yh(\boldsymbol{x}))] &\leq \mathbb{E}_{\mathcal{S}}[F_\phi(\hat{\pi}(\boldsymbol{x}), h(\boldsymbol{x}))] + \Lambda(\mathcal{S}) \ \text{(see (78) and (79))} \\
&\leq \mathbb{E}_{\mathcal{S}}[F_\phi(\hat{\pi}(\boldsymbol{x}), h(\boldsymbol{x}))] + \mathbb{E}_{\mathcal{D}_m} \sup_g \{\mathbb{E}_{\mathcal{D}}[F_\phi(yg(\boldsymbol{x}))] - \mathbb{E}_{\mathcal{S}}[F_\phi(\hat{\pi}(\boldsymbol{x}), g(\boldsymbol{x}))]\} \\
&\quad + Q_1 \sqrt{\frac{1}{2m} \log \frac{2}{\delta}} \ \text{(from (86))} \\
&\leq \mathbb{E}_{\mathcal{S}}[F_\phi(\hat{\pi}(\boldsymbol{x}), h(\boldsymbol{x}))] + \mathbb{E}_{\mathcal{D}_m, \mathcal{D}'_m} \sup_g \{\mathbb{E}_{\mathcal{S}'}[F_\phi(yg(\boldsymbol{x}))] - \mathbb{E}_{\mathcal{S}}[F_\phi(\hat{\pi}(\boldsymbol{x}), g(\boldsymbol{x}))]\} \\
&\quad + Q_1 \sqrt{\frac{1}{2m} \log \frac{2}{\delta}} \ \text{(from (87))} \\
&\leq \mathbb{E}_{\mathcal{S}}[F_\phi(\hat{\pi}(\boldsymbol{x}), h(\boldsymbol{x}))] + 2\mathbb{E}_{\mathcal{D}_m, \Sigma_m} \sup_g \{\mathbb{E}_{\mathcal{S}}[\sigma(\boldsymbol{x}) F_\phi(\hat{\pi}(\boldsymbol{x}), g(\boldsymbol{x}))]\} + L_{2m} \\
&\quad + Q_1 \sqrt{\frac{1}{2m} \log \frac{2}{\delta}} \ \text{(Lemma (7))} \\
&\leq \mathbb{E}_{\mathcal{S}}[F_\phi(\hat{\pi}(\boldsymbol{x}), h(\boldsymbol{x}))] + 2\mathbb{E}_{\Sigma_m} \sup_h \{\mathbb{E}_{\mathcal{S}}[\sigma(\boldsymbol{x}) F_\phi(\hat{\pi}(\boldsymbol{x}), h(\boldsymbol{x}))]\} + L_{2m} \\
&\quad + 2Q_1 \sqrt{\frac{1}{2m} \log \frac{2}{\delta}} \ \text{(from (107))} \\
&= \mathbb{E}_{\Sigma_{\hat{\boldsymbol{\pi}}}} \mathbb{E}_{\mathcal{S}}[F_\phi(\sigma(\boldsymbol{x}) h(\boldsymbol{x}))] + 2\hat{R}^b_m + L_{2m} + 4\left(\frac{2h_*}{b_\phi} + 1\right) \sqrt{\frac{1}{2m} \log \frac{2}{\delta}} \ ,
\end{aligned}$$

as claimed.

2.9.2 Proof of eq. (15)

We have $F'_\phi(x) = -(1/b_\phi))(\phi^\star)'(-x) = -(1/b_\phi)(\phi')^{-1}(-x) \in [-1/b_\phi, 0]$, and thus F_ϕ is $1/b_\phi$-Lipschitz, so Theorem 4.12 in [8] brings:

$$\begin{aligned}
R^b_m(F, \eta) &= \mathbb{E}_{\boldsymbol{\sigma} \sim \Sigma_m} \sup_{h \in \mathcal{H}} \left\{\mathbb{E}_{i \sim [m]}[\sigma_i \mathbb{E}_{\boldsymbol{\sigma}' \sim \Sigma_{\hat{\boldsymbol{\pi}}}}[F_\phi(\sigma'_i h(\boldsymbol{x}_i) - \eta)]]\right\} \\
&\leq b_\phi \mathbb{E}_{\boldsymbol{\sigma} \sim \Sigma_m} \sup_{h \in \mathcal{H}} \left\{\mathbb{E}_{i \sim [m]}[\sigma_i \mathbb{E}_{\boldsymbol{\sigma}' \sim \Sigma_{\hat{\boldsymbol{\pi}}}}[\sigma'_i h(\boldsymbol{x}_i) - \eta]]\right\} \\
&= b_\phi \mathbb{E}_{\boldsymbol{\sigma} \sim \Sigma_m} \sup_{h \in \mathcal{H}} \left\{\mathbb{E}_{i \sim [m]}[\sigma_i \mathbb{E}_{\boldsymbol{\sigma}' \sim \Sigma_{\hat{\boldsymbol{\pi}}}}[\sigma'_i h(\boldsymbol{x}_i)]]\right\} \\
&= b_\phi \mathbb{E}_{\boldsymbol{\sigma} \sim \Sigma_m} \sup_{h \in \mathcal{H}} \left\{\mathbb{E}_{i \sim [m]}[\sigma_i (2\hat{\pi}(\boldsymbol{x}_i) - 1) h(\boldsymbol{x}_i)]\right\} \ ,
\end{aligned}$$

as claimed.

3 Supplementary Material on Experiments

3.1 Full Experimental Setup

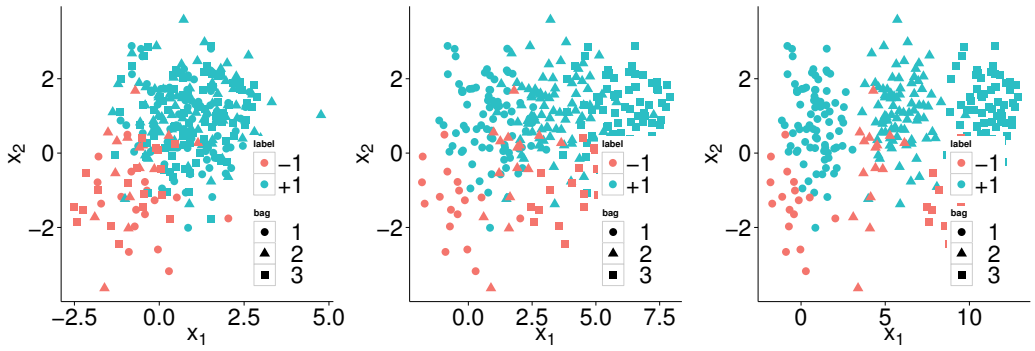
All mean operator algorithms have been coded in R. For $\propto$SVM and InvCal, we used a Matlab[1] implementation from the authors of [1]. The ranges of parameters for cross validation are $\lambda = \lambda' m$ with $\lambda' \in \{0\} \cup 10^{\{0,1,2\}}$, $\gamma \in 10^{-\{2,1,0\}}$, $\sigma \in 2^{-\{2,1,0\}}$ for mean operator algorithms. We ran all

[1] https:/github.com/felixyu/pSVM

experiments with $D_w = I$ and $\varepsilon = 0$. Since we tested on similar domains -6 are actually the same-ranges for InvCal and $\propto$SVM were taken from [1]. To avoid an additional source of complexity in the analysis, we cross-validated all hyper-parameters using the knowledge of all labels of the validation sets; notice that labels at validation time generally would not be accessible in real world applications.

3.2 Simulated Domain for Violation of Homogeneity Assumption

The synthetic data generated for this test consists on 16 classification problems, each one formed by 16 bags of 100 two-dimensional normal samples. The distribution generating the first dataset satisfies the homogeneity assumption (Figure 1 (a)). Then, we gradually change the position of the class-conditional bag-conditional means on one linear direction (to the right on Figure 1 (b) and (c)), with different offsets for different bags. In Figure 1 we give a graphical explanation of the process with 3 bags.



(a) (b) (c)

Figure 1: Violation of homogeneity assumption

3.3 Simulated Domain from [1]

The MM algorithm was shown to learn a model with zero accuracy prediction on the toy domain of [1]. We report here in Table 1 performance of all mean operator algorithms measured in transductive setting, training with cross-validation. Although none of the distances used in our experiments in LMM leads reasonable accuracy in the toy dataset, AMM^{max} initialised with *any* starting point learns *in one step* a model which perfectly classifies all the instances. We also notice that EMM returns an optimal classifier by itself (not reported in Table 1).

Table 1: AUC on the toy dataset of [1]

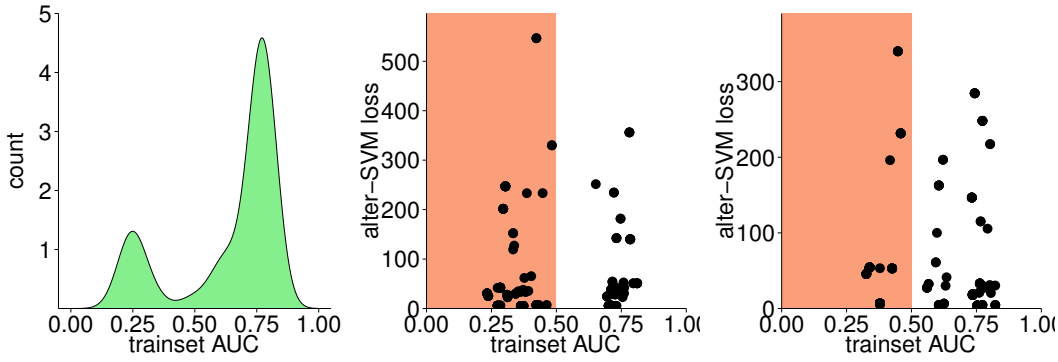
	AMM^{min}	AMM^{max}
EMM	100.00	100.00
MM	8.46	100.00
LMM_{G}	8.46	100.00
$\text{LMM}_{\text{G,s}}$	8.46	100.00
LMM_{nc}	8.46	100.00
1	8.46	100.00
10ran	100.00	100.00

3.4 Additional Tests on alter-$\propto$SVM [1]

In our experiments, we observe that AUC achieved by $\propto$SVM can be high, but it is also often *below* 0.5; in those cases the algorithm outputs models which are worse than random and the average performance over 5 test folds drops. We are able to reproduce the same behaviour on the *heart*

dataset provided by the authors in a demo for alter-$\propto$SVM; this also proves our bag assignment for LLP simulation does not introduce the issue. In a first test, we randomly select 3/4 of the dataset, and randomly assign instances to 4 bags of fixed size 64, following [1]. We repeat the training split 50 times with $C = C_p = 1$, as in the demo, and we measure AUCs on the same training set. As expected, a consistent number of run (22%) ends up producing AUC smaller than 0.5. We display in Figure 2 (a) the AUC's density profile, which shows a relevant mass around 0.25; notice also the two distribution modes look symmetric around 0.5.

In a second test, we investigate further measuring pairs of training set AUC and loss value obtained by the same execution of the algorithm. In this case, we run over all parameters ranges defined in $\propto$SVM's paper, and do not pick the model that minimizes the loss over the 10 random runs, but record losses of all. Figures 2 (b) and (c) show scatter plots relative to two chosen training set splits. We observe that loss minimization can lead both to high and low AUCs, with only few points close to 0.5. A possible explanation might be in the inverted polarity of the learnt linear classifier; inverted polarity in this contest means having a model which would achieve better performance classifying instances labels opposite to the ones predicted. We conclude that optimizing $\propto$SVM's loss in some cases might be equivalent to train a max-margin separator of the unlabelled data, which only exploits weakly the information given by the label proportions. This would give a heuristic understanding of the frequent symmetrical behaviour of the AUC.



(a) (b) (c)

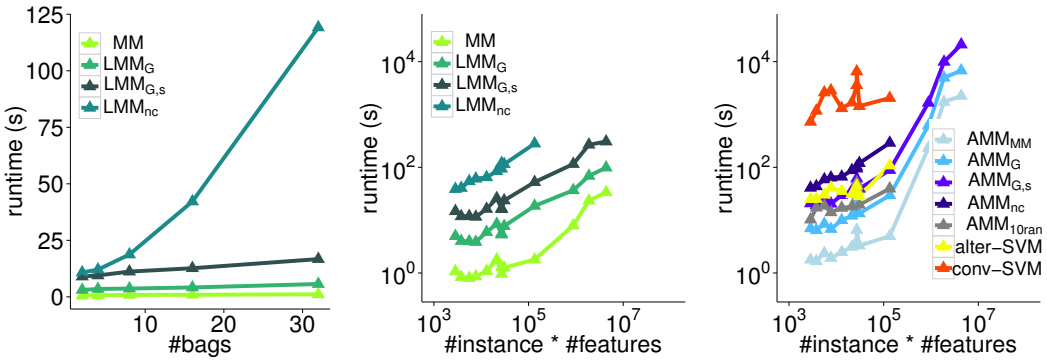
Figure 2: alter-$\propto$SVM: empirical distribution of AUC (a), and relationship between loss and AUC in two different train spit (b)(c)

3.5 Scalability

Figure 3 (a) shows runtime of learning (including cross-validation) of MM and LMM with regard to the number of bags – which is the natural parameter of time complexity for our Laplacian-based methods. Although the 3 layers of cross-validation of $\text{LMM}_{\text{G,s}}$, LMM_{nc} results the only method clearly not scalable. Figure 3 (b) presents how our one-shots algorithms scale on all small domains as a function of problem size. Runtime is averaged over the different bag assignments. The same plot is given in Figure 3 (c) for iterative algorithms, in particular AMM^{min} and (alter/conv)-$\propto$SVM. All curves are completed with measurements on bigger domains when available. Runtime of SVMs is not directly comparable with our methods. This is due to both (a) the implementation on different programming languages and (b) to the fact that the code provided implements kernel SVM, even for linear kernels, which is a big overhead in computation and memory access. Nevertheless, the high growth rate of conv-$\propto$SVM makes the algorithm not suitable for large datasets. Noticeably, even if alter-$\propto$SVM does not show such behaviour, we are not able to run it on our bigger domains, since it requires approximately 10 hours to run on a training set split with fixed parameters.

3.6 Full Results on Small Domains

Finally we report details about all experiments run on the 10 small domains (Table 2). In the following Tables, columns show the number of bags generated through K-MEANS. Each cell contains



(a) (b) (c)

Figure 3: Learning runtime of LMM for bags number (a), and for domain size one-shot (b) and iterative methods (c)

Table 2: Small domains size

dataset	instances	feature
arrhythmia	452	297
australian	690	39
breastw	699	11
colic	368	83
german	1000	27
heart	270	14
ionosphere	351	37
vertebral column	620	9
vote	435	49
wine	178	16

average AUC over 5 test splits and standard deviation; runtime in second is in the separated column. Best performing algorithm and ones not worse than 0.1 AUC are bold faced. Comparisons are made in the respective top/bottom sub-tables, which group one-shot and iterative algorithms. We use ↑ to highlight runs which achieve average AUC greater or equal than the Oracle.

Table 3: *arrhythmia*

algorithm		2 bags AUC	2 bags time(s)	4 bags AUC	4 bags time(s)	8 bags AUC	8 bags time(s)	16 bags AUC	16 bags time(s)	32 bags AUC	32 bags time(s)
	EMM	**70.91 ± 6.81**	2	50.55 ± 7.54	2	50.31 ± 7.55	2	47.03 ± 6.60	2	52.34 ± 7.25	2
	MM	64.99 ± 2.99	2	60.48 ± 7.28	1	68.17 ± 5.95	2	70.01 ± 9.33	2	72.85 ± 9.49	2
	LMM_G	64.99 ± 2.99	18	68.10 ± 4.43	17	**71.53 ± 2.36**	20	**72.06 ± 7.62**	18	**76.29 ± 7.91**	20
	$LMM_{G,s}$	64.99 ± 2.99	49	**68.34 ± 3.95**	49	**71.53 ± 2.36**	54	**72.06 ± 7.62**	52	**76.29 ± 7.91**	57
	LMM_{nc}	64.99 ± 2.99	83	61.19 ± 7.53	83	70.21 ± 5.17	119	70.89 ± 9.86	267	73.82 ± 9.29	854
	InvCal	64.75 ± 3.04	17	66.12 ± 260	17	60.87 ± 3.54	17	44.46 ± 3.36	17	56.36 ± 5.26	17
AMM^{min}	AMM_{EMM}	59.54 ± 7.52	9	52.65 ± 3.10	8	63.46 ± 10.37	8	67.85 ± 9.56	8	75.65 ± 8.81	8
AMM^{min}	AMM_{MM}	57.29 ± 5.95	7	60.00 ± 7.96	4	70.12 ± 6.46	4	73.66 ± 8.86	5	78.36 ± 8.53	5
AMM^{min}	AMM_G	58.15 ± 6.83	31	68.80 ± 2.15	28	**73.08 ± 2.92**	30	74.54 ± 7.98	29	**80.32 ± 8.08**	30
AMM^{min}	$AMM_{G,s}$	56.67 ± 4.66	92	69.83 ± 2.69	84	**73.08 ± 2.92**	88	73.34 ± 7.62	88	**80.32 ± 8.08**	91
AMM^{min}	AMM_{nc}	57.29 ± 5.95	97	59.71 ± 8.39	90	71.43 ± 6.21	126	73.49 ± 8.95	274	78.04 ± 8.26	862
AMM^{min}	AMM_1	**65.80 ± 6.92**	5	**70.00 ± 5.89**	4	68.17 ± 7.19	4	69.93 ± 4.27	4	72.31 ± 5.02	5
AMM^{min}	AMM_{10ran}	54.09 ± 12.03	30	55.78 ± 17.36	32	66.38 ± 7.32	51	66.89 ± 6.75	51	73.61 ± 5.15	57
AMM^{max}	AMM_{EMM}	50.59 ± 5.97	41	59.32 ± 5.82	41	60.85 ± 5.43	37	60.38 ± 4.08	41	58.31 ± 8.40	40
AMM^{max}	AMM_{MM}	62.08 ± 9.46	45	46.86 ± 3.90	34	67.28 ± 8.92	33	74.04 ± 9.46	35	71.00 ± 7.65	38
AMM^{max}	AMM_G	62.08 ± 9.46	141	62.27 ± 8.14	128	65.78 ± 3.92	118	64.64 ± 10.26	121	73.07 ± 6.72	124
AMM^{max}	$AMM_{G,s}$	62.08 ± 9.46	414	63.13 ± 5.17	380	63.85 ± 7.00	346	65.49 ± 10.62	354	73.05 ± 6.70	374
AMM^{max}	AMM_{nc}	62.08 ± 9.46	206	55.57 ± 6.07	182	64.30 ± 6.24	207	**76.33 ± 3.96**	362	70.82 ± 4.23	965
AMM^{max}	AMM_1	60.53 ± 9.79	31	54.14 ± 13.28	34	67.45 ± 3.91	32	55.85 ± 8.96	35	61.26 ± 6.95	38
AMM^{max}	AMM_{10ran}	49.79 ± 8.14	307	55.37 ± 14.62	370	53.78 ± 5.13	301	60.62 ± 8.04	322	64.20 ± 2.84	338
SVM	alter-$\propto$	49.24 ± 3.92	96	57.10 ± 2.71	100	56.38 ± 2.73	104	35.31 ± 1.30	114	38.68 ± 6.10	125
SVM	conv-$\propto$	54.15 ± 2.22	2054	34.82 ± 3.20	2078	38.31 ± 8.24	2168	61.96 ± 1.10	1930	48.77 ± 5.73	2004
	Oracle	99.99 ± 0.02	2	99.98 ± 0.05	2	99.94 ± 0.13	2	100.00 ± 0.00	2	99.97 ± 0.07	2

Table 4: *australian*

algorithm		2 bags AUC	2 bags time(s)	4 bags AUC	4 bags time(s)	8 bags AUC	8 bags time(s)	16 bags AUC	16 bags time(s)	32 bags AUC	32 bags time(s)
	EMM	66.48 ± 3.16	<1	64.67 ± 4.22	<1	63.56 ± 4.00	<1	64.17 ± 4.80	<1	63.14 ± 5.41	<1
	MM	**81.08 ± 1.66**	<1	87.11 ± 2.68	<1	87.49 ± 2.86	1	87.36 ± 2.22	<1	89.53 ± 2.13	2
	LMM_G	**81.08 ± 1.66**	4	87.09 ± 2.82	4	**87.81 ± 3.16**	5	88.46 ± 2.50	6	89.69 ± 2.68	8
	$LMM_{G,s}$	**81.08 ± 1.66**	14	**87.81 ± 3.08**	15	**87.88 ± 3.21**	19	**89.18 ± 2.05**	20	**90.80 ± 2.53**	27
	LMM_{nc}	**81.08 ± 1.66**	57	87.02 ± 2.72	49	87.46 ± 3.03	57	88.06 ± 2.31	90	89.41 ± 2.41	217
	Invcal	19.67 ± 2.23	5	59.50 ± 5.86	5	68.00 ± 5.27	5	60.83 ± 3.17	5	51.81 ± 4.72	5
AMM^{min}	AMM_{EMM}	86.65 ± 2.06	4	**86.59 ± 3.08**	4	86.50 ± 4.11	4	89.51 ± 2.48	6	88.85 ± 4	6
AMM^{min}	AMM_{MM}	**87.54 ± 3.84**	3	84.35 ± 3.63	4	**86.99 ± 3.87**	4	89.43 ± 1.34	4	89.55 ± 3.18	5
AMM^{min}	AMM_G	**87.54 ± 3.84**	10	84.79 ± 3.17	13	86.78 ± 4.21	14	89.52 ± 2.18	14	89.88 ± 2.78	18
AMM^{min}	$AMM_{G,s}$	**87.54 ± 3.84**	30	85.12 ± 3.75	39	86.75 ± 4.19	43	**90.37 ± 1.67**	43	89.95 ± 2.80	54
AMM^{min}	AMM_{nc}	**87.54 ± 3.84**	63	85.10 ± 3.55	57	86.63 ± 4.02	66	89.00 ± 1.83	97	**90.11 ± 2.93**	227
AMM^{min}	AMM_1	72.60 ± 5.70	2	85.04 ± 2.53	3	**86.89 ± 3.73**	4	88.91 ± 2.32	4	88.98 ± 3.00	4
AMM^{min}	AMM_{10ran}	79.21 ± 5.07	27	80.97 ± 2.27	31	85.08 ± 3.30	34	89.19 ± 1.81	46	87.70 ± 2.68	47
AMM^{max}	AMM_{EMM}	80.09 ± 3.99	17	71.46 ± 1.85	16	73.41 ± 6.07	16	73.25 ± 3.33	18	81.73 ± 3.60	19
AMM^{max}	AMM_{MM}	86.83 ± 4.26	20	72.96 ± 2.30	15	70.25 ± 4.65	16	73.89 ± 5.77	18	75.91 ± 3.50	21
AMM^{max}	AMM_G	86.83 ± 4.26	61	73.32 ± 1.95	48	71.16 ± 4.94	51	73.57 ± 6.86	55	75.25 ± 3.18	63
AMM^{max}	$AMM_{G,s}$	86.83 ± 4.26	181	73.25 ± 2.03	143	71.19 ± 4.91	153	74.77 ± 6.85	163	75.25 ± 3.18	188
AMM^{max}	AMM_{nc}	86.83 ± 4.26	114	73.74 ± 2.48	92	70.36 ± 5.16	102	75.16 ± 5.71	138	76.44 ± 2.74	272
AMM^{max}	AMM_1	69.57 ± 3.99	15	73.12 ± 3.41	15	68.25 ± 2.80	16	71.02 ± 5.46	17	81.70 ± 3.02	19
AMM^{max}	AMM_{10ran}	77.82 ± 9.12	192	68.82 ± 4.73	138	73.58 ± 4.29	146	72.21 ± 9.35	164	74.16 ± 5.25	188
SVM	alter-$\propto$	53.26 ± 2.07	25	51.08 ± 2.35	27	50.90 ± 1.63	31	48.29 ± 4.51	38	41.66 ± 5.11	64
SVM	conv-$\propto$	77.80 ± 6.16	3924	66.14 ± 4.68	3790	57.94 ± 18.54	3244	61.37 ± 21.17	3327	63.73 ± 11.33	3603
	Oracle	92.81 ± 2.89	<1	92.68 ± 2.24	<1	92.44 ± 3.01	,1	92.61 ± 2.03	<1	92.99 ± 3.58	<1

Table 5: *breastw*

algorithm		2 bags AUC	2 bags time(s)	4 bags AUC	4 bags time(s)	8 bags AUC	8 bags time(s)	16 bags AUC	16 bags time(s)	32 bags AUC	32 bags time(s)
	EMM	48.65 ± 7.54	<1	71.45 ± 16.59	<1	61.68 ± 7.47	<1	34.88 ± 12.33	<1	47.50 ± 22.77	<1
	MM	**99.42 ± 0.44**	2	**99.30 ± 0.39**	<1	**99.28 ± 0.25**	<1	**99.28 ± 0.37**	<1	**99.18 ± 0.47**	1
	LMM_G	**99.42 ± 0.44**	6	**99.33 ± 0.38**	3	**99.28 ± 0.25**	3	**99.35 ± 0.39**	3	**99.22 ± 0.46**	4
	$LMM_{G,s}$	**99.42 ± 0.44**	20	**99.34 ± 0.39**	10	**99.37 ± 0.24** ↑	11	**99.36 ± 0.38**	12	**99.23 ± 0.44**	15
	LMM_{nc}	**99.42 ± 0.44**	41	**99.29 ± 0.40**	39	**99.27 ± 0.25**	41	**99.30 ± 0.38**	59	**99.20 ± 0.47**	125
	Invcal	19.67 ± 2.23	5	59.50 ± 5.86	5	68 ± 5.27	5	60.83 ± 3.17	5	51.81 ± 4.72	5
AMM^{min}	AMM_{EMM}	**99.37 ± 0.42**	1	**99.33 ± 0.39**	1	99.17 ± 0.54	1	**99.34 ± 0.40**	2	99.29 ± 0.49	2
AMM^{min}	AMM_{MM}	**99.34 ± 0.46**	2	**99.30 ± 0.37**	1	**99.36 ± 0.27** ↑	2	99.29 ± 0.41	2	99.29 ± 0.48	2
AMM^{min}	AMM_G	**99.34 ± 0.46**	8	**99.30 ± 0.37** ↑	5	**99.36 ± 0.27** ↑	6	99.29 ± 0.41	7	99.30 ± 0.49	8
AMM^{min}	$AMM_{G,s}$	**99.34 ± 0.46**	23	**99.30 ± 0.37** ↑	16	**99.36 ± 0.27** ↑	19	99.29 ± 0.41	20	99.30 ± 0.49	25
AMM^{min}	AMM_{nc}	**99.34 ± 0.46**	43	**99.31 ± 0.35**	41	**99.36 ± 0.27** ↑	44	99.29 ± 0.41	62	99.29 ± 0.48	129
AMM^{min}	AMM_1	**99.35 ± 0.45**	<1	**99.32 ± 0.37**	1	99.20 ± 0.45	1	99.30 ± 0.42	1	**99.31 ± 0.48**	2
AMM^{min}	AMM_{10ran}	**99.36 ± 0.45**	8	99.11 ± 0.56	9	99.26 ± 0.35	11	99.28 ± 0.43	11	**99.32 ± 0.49** ↑	14
AMM^{max}	AMM_{EMM}	**99.42 ± 0.55**	6	99.02 ± 0.66	6	**99.32 ± 0.25** ↑	6	**99.43 ± 0.30** ↑	7	**99.40 ± 0.38** ↑	9
AMM^{max}	AMM_{MM}	99.01 ± 1.12	6	99.00 ± 0.64	6	**99.32 ± 0.35** ↑	6	**99.37 ± 0.38**	7	**99.39 ± 0.39** ↑	9
AMM^{max}	AMM_G	99.01 ± 1.12	20	98.99 ± 0.64	17	**99.33 ± 0.35** ↑	18	**99.37 ± 0.38**	21	**99.41 ± 0.39** ↑	27
AMM^{max}	$AMM_{G,s}$	99.01 ± 1.12	60	98.99 ± 0.64	52	99.19 ± 0.45	55	**99.37 ± 0.39**	63	**99.41 ± 0.39** ↑	82
AMM^{max}	AMM_{nc}	99.01 ± 1.12	55	98.99 ± 0.64	53	**99.32 ± 0.35** ↑	56	**99.37 ± 0.39**	76	**99.40 ± 0.38** ↑	148
AMM^{max}	AMM_1	99.09 ± 1.08	5	99.09 ± 0.46	5	**99.29 ± 0.26**	5	**99.37 ± 0.38**	6	**99.40 ± 0.38** ↑	8
AMM^{max}	AMM_{10ran}	98.97 ± 1.29	47	98.58 ± 0.75	48	**99.39 ± 0.27** ↑	52	**99.37 ± 0.38**	61	**99.36 ± 0.41** ↑	81
SVM	alter-$\propto$	68.63 ± 17.63	24	93.24 ± 4.43	25	75.17 ± 7.19	33	90.11 ± 2.58	42	18.23 ± 5.67	82
SVM	conv-$\propto$	**99.41 ± 0.48**	3346	56.33 ± 4.28	3043	77.71 ± 15.51	2800	32.90 ± 7.24	3036	67.21 ± 8.19	2037
	Oracle	99.48 ± 0.41	<1	99.53 ± 0.41	<1	99.31 ± 0.37	<1	99.43 ± 0.39	<1	99.32 ± 0.44	<1

Table 6: *colic*

algorithm		2 bags AUC	2 bags time(s)	4 bags AUC	4 bags time(s)	8 bags AUC	8 bags time(s)	16 bags AUC	16 bags time(s)	32 bags AUC	32 bags time(s)
	EMM	60.69 ± 11.30	<1	51.83 ± 6.36	<1	52.99 ± 5.37	<1	53.83 ± 11.49	<1	52.95 ± 13.28	<1
	MM	**62.00 ± 6.44**	<1	70.48 ± 7.43	<1	67.13 ± 9.85	2	72.60 ± 9.35	1	72.05 ± 3.38	1
	LMM_G	**62.00 ± 6.44**	7	70.37 ± 7.47	6	72.15 ± 8.51	8	75.96 ± 10.38	8	75.47 ± 3.59	9
	$LMM_{G,s}$	**62.00 ± 6.44**	20	**72.10 ± 6.26**	20	**75.08 ± 7.14**	28	**78.54 ± 10.20**	26	**76.43 ± 3.10**	27
	LMM_{nc}	**62.00 ± 6.44**	31	70.45 ± 7.46	33	68.38 ± 9.69	52	74.04 ± 10.02	112	72.87 ± 3.20	345
	Invcal	38.73 ± 5.43	6	65.87 ± 6.70	6	59.30 ± 3.28	6	61.54 ± 4.17	6	59.53 ± 10.00	6
AMM^{min}	AMM_{EMM}	59.12 ± 8.86	3	56.23 ± 8.49	3	70.93 ± 10.31	3	**78.22 ± 6.00**	3	74.22 ± 6.35	4
AMM^{min}	AMM_{MM}	**77.44 ± 3.16**	2	78.84 ± 6.95	3	69.46 ± 6.44	4	71.93 ± 7.61	4	81.44 ± 5.18	4
AMM^{min}	AMM_G	**77.44 ± 3.16**	11	**79.41 ± 2.23**	12	72.62 ± 5.42	14	77.80 ± 8.11	14	**84.05 ± 2.33**	16
AMM^{min}	$AMM_{G,s}$	**77.44 ± 3.16**	34	**79.41 ± 2.23**	36	71.19 ± 5.38	41	76.71 ± 6.70	40	83.27 ± 3.14	47
AMM^{min}	AMM_{nc}	**77.44 ± 3.16**	36	78.33 ± 7.35	38	70.95 ± 4.69	57	74.67 ± 9.10	117	79.86 ± 4.87	352
AMM^{min}	AMM_1	38.69 ± 7.18	1	56.07 ± 14.68	2	**75.14 ± 4.78**	2	75.36 ± 5.64	3	77.51 ± 5.00	3
AMM^{min}	AMM_{10ran}	37.63 ± 4.19	10	77.75 ± 5.66	12	74.95 ± 5.64	15	76.59 ± 10.81	17	78.94 ± 4.17	23
AMM^{max}	AMM_{EMM}	50.94 ± 6.54	9	62.44 ± 9.94	9	57.53 ± 13.37	15	53.63 ± 14.71	17	67.63 ± 5.63	19
AMM^{max}	AMM_{MM}	43.05 ± 14.65	8	75.40 ± 4.64	9	63.72 ± 14.41	16	55.37 ± 10.19	18	69.49 ± 3.17	20
AMM^{max}	AMM_G	43.05 ± 14.65	28	78.19 ± 5.93	31	63.14 ± 7.53	51	61.32 ± 5.69	57	68.21 ± 9.35	62
AMM^{max}	$AMM_{G,s}$	43.05 ± 14.65	84	77.91 ± 6.36	91	62.57 ± 6.11	151	64.42 ± 10.77	168	69.47 ± 6.40	184
AMM^{max}	AMM_{nc}	42.92 ± 14.74	52	73.74 ± 7.21	57	60.39 ± 12.21	94	62.46 ± 15.13	162	68.63 ± 2.37	381
AMM^{max}	AMM_1	51.92 ± 19.91	7	59.89 ± 10.79	8	58.76 ± 12.16	14	62.31 ± 13.32	17	68.25 ± 6.42	18
AMM^{max}	AMM_{10ran}	56.39 ± 10.26	60	71.28 ± 8.76	68	65.01 ± 13.85	114	69.59 ± 9.96	139	74.40 ± 5.54	159
SVM	alter-∝	46.33 ± 2.73	18	50.82 ± 1.21	19	60.84 ± 5.51	23	62.20 ± 3.79	32	57.04 ± 10.10	49
SVM	conv-∝	25.27 ± 3.45	1438	35.96 ± 9.34	1460	50.31 ± 5.57	1439	35.46 ± 9.11	1423	50.13 ± 8.34	1427
	Oracle	86.19 ± 4.23	<1	87.80 ± 2.50	<1	87.05 ± 6.05	<1	86.53 ± 7.15	<1	87.97 ± 2.02	<1

Table 7: *german*

algorithm		2 bags AUC	2 bags time(s)	4 bags AUC	4 bags time(s)	8 bags AUC	8 bags time(s)	16 bags AUC	16 bags time(s)	32 bags AUC	32 bags time(s)
	EMM	47.90 ± 4.51	<1	50.11 ± 5.17	<1	46.02 ± 5.88	<1	50.94 ± 1.61	<1	51.02 ± 2.55	<1
	MM	**61.07 ± 5.57**	<1	62.09 ± 4.00	<1	65.50 ± 6.54	2	65.61 ± 6.05	2	66.96 ± 4.56	2
	LMM_G	**61.07 ± 5.57**	4	62.14 ± 4.04	4	67.07 ± 6.36	6	66.43 ± 6.61	6	70.18 ± 4.76	7
	$LMM_{G,s}$	**61.07 ± 5.57**	11	62.75 ± 3.32	12	**67.91 ± 5.80**	16	**66.40 ± 6.90**	19	**70.43 ± 5.57**	21
	LMM_{nc}	**61.07 ± 5.57**	103	62.04 ± 4.00	87	65.47 ± 6.56	87	65.61 ± 6.06	113	67.01 ± 4.58	209
	Invcal	38.74 ± 5.43	6	**65.87 ± 6.70**	6	59.30 ± 3.28	6	61.53 ± 4.17	6	59.54 ± 10.00	6
AMM^{min}	AMM_{EMM}	53.89 ± 6.82	7	48.63 ± 8.71	7	53.24 ± 8.02	8	57.58 ± 3.44	9	63.64 ± 11.82	11
AMM^{min}	AMM_{MM}	**60.45 ± 5.58**	5	63.33 ± 4.99	6	74.58 ± 4.76	6	72.43 ± 1.39	8	75.84 ± 5.24	7
AMM^{min}	AMM_G	**60.45 ± 5.58**	17	**64.16 ± 6.99**	18	74.18 ± 4.34	21	72.08 ± 1.24	22	**75.94 ± 4.55**	24
AMM^{min}	$AMM_{G,s}$	**60.45 ± 5.58**	52	**64.20 ± 7.24**	57	74.29 ± 4.50	57	72.18 ± 1.37	66	75.77 ± 4.44	74
AMM^{min}	AMM_{nc}	**60.45 ± 5.58**	118	63.20 ± 6.09	101	**75.37 ± 4.42**	100	72.53 ± 1.25	130	**75.99 ± 5.26**	225
AMM^{min}	AMM_1	37.08 ± 4.42	3	38.53 ± 2.97	3	41.89 ± 2.07	6	41.13 ± 2.58	9	47.09 ± 9.40	10
AMM^{min}	AMM_{10ran}	49.12 ± 6.50	36	60.31 ± 5.57	38	73.82 ± 4.70	44	72.07 ± 3.22	54	74.73 ± 4.54	72
AMM^{max}	AMM_{EMM}	46.45 ± 3.30	18	46.31 ± 3.02	19	67.34 ± 13.42	19	72.41 ± 6.17	20	74.58 ± 4.63	22
AMM^{max}	AMM_{MM}	52.47 ± 8.88	18	58.61 ± 12.19	18	65.14 ± 21.84	19	74.90 ± 4.86	20	74.88 ± 3.75	22
AMM^{max}	AMM_G	52.47 ± 8.88	54	56.12 ± 12.25	53	74.93 ± 8.18	57	73.87 ± 4.55	60	75.43 ± 4.02	67
AMM^{max}	$AMM_{G,s}$	52.47 ± 8.88	160	54.79 ± 11.61	158	74.84 ± 8.12	167	73.87 ± 4.55	180	75.40 ± 4.05	197
AMM^{max}	AMM_{nc}	52.47 ± 8.88	154	49.24 ± 12.68	137	65.11 ± 21.84	137	74.89 ± 4.75	167	74.70 ± 3.71	269
AMM^{max}	AMM_1	58.39 ± 13.20	17	61.04 ± 14.43	17	69.66 ± 16.93	17	**76.49 ± 3.29**	18	75.44 ± 3.65	20
AMM^{max}	AMM_{10ran}	50.47 ± 9.69	168	56.78 ± 10.89	164	60.41 ± 15.48	160	61.62 ± 18.81	170	73.25 ± 6.97	191
SVM	alter-∝	49.36 ± 1.68	34	49.59 ± 1.58	37	48.43 ± 2.23	40	48.85 ± 1.55	47	51.05 ± 2.72	64
SVM	conv-∝	29.70 ± 2.03	6031	**64.15 ± 5.43**	6343	63.01 ± 2.59	6362	62.01 ± 3.61	6765	63.17 ± 3.62	7004
	Oracle	79.43 ± 2.88	<1	78.95 ± 3.99	<1	79.18 ± 1.70	<1	79.42 ± 2.80	<1	79.02 ± 3.62	<1

Table 8: *heart*

algorithm		2 bags AUC	2 bags time(s)	4 bags AUC	4 bags time(s)	8 bags AUC	8 bags time(s)	16 bags AUC	16 bags time(s)	32 bags AUC	32 bags time(s)
	EMM	51.82 ± 12.39	<1	50.43 ± 23.03	<1	55.09 ± 19.44	<1	49.55 ± 17.47	<1	63.49 ± 18.11	<1
	MM	**68.75 ± 6.09**	<1	60.24 ± 13.54	<1	80.35 ± 9.42	<1	76.11 ± 6.66	1	83.50 ± 6.22	1
	LMM_G	**68.75 ± 6.09**	3	68.04 ± 8.53	3	82.87 ± 6.16	4	**82.92 ± 1.28**	4	85.85 ± 3.84	6
	$LMM_{G,s}$	**68.75 ± 6.09**	9	**69.04 ± 6.52**	12	**83.68 ± 5.90**	13	**82.96 ± 1.79**	14	**86.36 ± 3.94**	17
	LMM_{nc}	**68.75 ± 6.09**	11	60.40 ± 14.18	12	80.24 ± 9.74	189	78.14 ± 4.98	42	84.47 ± 5.06	119
	Invcal	28.84 ± 4.96	4	70.58 ± 6.45	4	37.33 ± 10.31	4	44.96 ± 9.64	4	62.76 ± 15.05	4
AMM^{min}	AMM_{EMM}	60.50 ± 30.88	<1	63.36 ± 28.50	1	72.05 ± 19.17	1	80.87 ± 15.51	1	**91.63 ± 6.10** ↑	2
AMM^{min}	AMM_{MM}	86.59 ± 6.14	1	80.57 ± 16.72	1	87.96 ± 4.50	2	90.04 ± 5.14	2	91.45 ± 5.70 ↑	2
AMM^{min}	AMM_G	86.59 ± 6.14	5	86.70 ± 5.45	5	87.46 ± 2.67	6	**91.06 ± 2.87**	7	**91.55 ± 5.93** ↑	9
AMM^{min}	$AMM_{G,s}$	86.59 ± 6.14	15	86.70 ± 5.45	16	88.31 ± 4.00	18	90.86 ± 2.81	21	**91.55 ± 5.93** ↑	27
AMM^{min}	AMM_{nc}	86.59 ± 6.14	13	78.97 ± 16.78	14	87.82 ± 4.42	21	90.48 ± 3.53	45	91.25 ± 5.77	125
AMM^{min}	AMM_1	**90.62 ± 5.82**	<1	89.19 ± 5.90	1	88.64 ± 3.21	1	90.78 ± 2.10	1	91.03 ± 5.82	1
AMM^{min}	AMM_{10ran}	78.38 ± 30.44	5	87.32 ± 4.71	6	89.85 ± 2.31	7	**91.02 ± 2.49**	9	90.47 ± 6.39	14
AMM^{max}	AMM_{EMM}	85.74 ± 13.28	3	84.60 ± 10.87	4	84.60 ± 7.84	3	89.83 ± 2.72	5	71.65 ± 18.52	6
AMM^{max}	AMM_{MM}	85.35 ± 11.06	4	82.43 ± 9.76	4	**90.49 ± 4.75**	4	89.92 ± 2.90		89.35 ± 6.98	7
AMM^{max}	AMM_G	85.35 ± 11.06	13	87.18 ± 6.56	13	**90.49 ± 4.75**	13	89.58 ± 2.79	16	88.55 ± 9.71	23
AMM^{max}	$AMM_{G,s}$	85.35 ± 11.06	39	**90.49 ± 5.05**	40	**90.58 ± 4.77**	40	89.58 ± 2.79	49	89.94 ± 6.63	67
AMM^{max}	AMM_{nc}	85.35 ± 11.06	20	82.73 ± 9.23	21	89.84 ± 4.24	30	90.06 ± 3.20	54	89.54 ± 6.60	140
AMM^{max}	AMM_1	72.77 ± 37.27	4	89.31 ± 3.99	3	89.68 ± 3.79	3	90.62 ± 3.18	5	87.97 ± 9.42	6
AMM^{max}	AMM_{10ran}	89.96 ± 5.62	32	89.93 ± 5.02	31	88.03 ± 3.16	30	90.80 ± 3.61	38	89.61 ± 8.68	54
SVM	alter-∝	47.75 ± 17.58	15	59.72 ± 18.21	16	62.32 ± 12.83	20	58.49 ± 10.98	27	48.33 ± 12.77	47
SVM	conv-∝	46.18 ± 43.41	1211	87.13 ± 5.30	1185	69.03 ± 23.18	1197	42.78 ± 23.51	1188	50.34 ± 15.75	1080
	Oracle	91.72 ± 3.95	<1	91.22 ± 4.09	<1	91.27 ± 2.88	<1	91.54 ± 2.76	<1	91.42 ± 5.46	<1

Table 9: *ionosphere*

algorithm		2 bags AUC	2 bags time(s)	4 bags AUC	4 bags time(s)	8 bags AUC	8 bags time(s)	16 bags AUC	16 bags time(s)	32 bags AUC	32 bags time(s)
	EMM	44.28 ± 12.13	<1	51.86 ± 8.01	<1	50.69 ± 6.34	<1	44.60 ± 3.91	<1	48.91 ± 11.73	<1
	MM	**64.81 ± 8.82**	<1	77.74 ± 5.23	1	78.95 ± 7.36	1	86.76 ± 2.96	1	88.13 ± 4.16	2
	LMM_G	**64.81 ± 8.82**	5	80.80 ± 2.32	6	**83.46 ± 4.62**	5	87.12 ± 2.23	7	**88.24 ± 4.41**	7
	$LMM_{G,s}$	**64.81 ± 8.82**	14	**82.12 ± 2.50**	15	83.24 ± 4.84	15	**87.23 ± 1.57**	17	87.99 ± 4.58	21
	LMM_{nc}	**64.81 ± 8.82**	20	79.39 ± 2.12	22	81.18 ± 6.40	32	87.05 ± 2.48	68	**88.34 ± 4.32**	182
	Invcal	35.34 ± 8.76	5	44.78 ± 15.37	5	53.28 ± 9.02	5	53.52 ± 8.51	5	54.08 ± 9.53	5
AMM^{min}	AMM_{EMM}	56.77 ± 6.42	2	**85.07 ± 5.24**	2	86.04 ± 5.21	2	86.81 ± 3.81	2	86.71 ± 3.54	3
AMM^{min}	AMM_{MM}	46.67 ± 8.53	3	84.52 ± 4.60	2	84.23 ± 6.67	2	85.92 ± 4.48	3	87.77 ± 5.56	3
AMM^{min}	AMM_G	46.67 ± 8.53	10	85.05 ± 4.11	9	85.28 ± 6.19	9	85.97 ± 3.19	11	88.85 ± 5.15	12
AMM^{min}	$AMM_{G,s}$	46.67 ± 8.53	28	84.63 ± 3.80	26	85.28 ± 6.19	27	86.01 ± 4.37	30	88.85 ± 5.15	36
AMM^{min}	AMM_{nc}	46.67 ± 8.53	24	**85.16 ± 4.39**	26	84.77 ± 6.45	36	85.96 ± 4.50	72	87.57 ± 5.23	174
AMM^{min}	AMM_1	51.47 ± 13.46	1	83.65 ± 3.89	2	**87.51 ± 4.24**	2	86.76 ± 4.07	2	87.83 ± 5.05	2.11
AMM^{min}	AMM_{10ran}	56.92 ± 22.42	10	80.39 ± 6.36	11	85.89 ± 5.52	12	87.32 ± 3.17	13	87.81 ± 6.52	15
AMM^{max}	AMM_{EMM}	57.99 ± 8.96	10	76.31 ± 5.29	10	82.07 ± 4.47	11	86.99 ± 7.23	11	87.08 ± 5.86	12
AMM^{max}	AMM_{MM}	**74.57 ± 18.16**	10	75.32 ± 4.74	10	78.65 ± 7.93	11	88.84 ± 3.10	12	**90.01 ± 5.50**	13
AMM^{max}	AMM_G	**74.57 ± 18.16**	32	78.06 ± 5.11	33	83.24 ± 6.54	35	89.98 ± 3.08 ↑	38	88.41 ± 5.94	41
AMM^{max}	$AMM_{G,s}$	**74.57 ± 18.16**	96	79.21 ± 4.58	98	83.36 ± 6.61	104	**90.88 ± 3.11** ↑	112	88.41 ± 5.94	121
AMM^{max}	AMM_{nc}	**74.57 ± 18.16**	47	75.80 ± 5.14	50	80.22 ± 6.95	61	88.05 ± 2.47	99	89.19 ± 5.45	198
AMM^{max}	AMM_1	65.53 ± 17.30	10	77.29 ± 6.63	9	82.10 ± 7.95	10	85.45 ± 3.31	11	89.01 ± 7.02	12
AMM^{max}	AMM_{10ran}	65.05 ± 16.59	85	79.60 ± 6.56	82	78.56 ± 4.77	88	88.44 ± 3.22	94	89.37 ± 6.67	109
SVM	alter-∝	43.07 ± 6.05	22	44.58 ± 4.95	24	69.24 ± 4.99	27	67.72 ± 12.25	55	59.67 ± 7.01	49
SVM	conv-∝	36.67 ± 7.44	1316	44.55 ± 9.58	1280	57.84 ± 5.98	1788	65.93 ± 3.90	887	47.58 ± 11.29	1287
	Oracle	90.07 ± 5.04	<1	89.99 ± 4.23	<1	90.08 ± 5.50	<1	89.42 ± 6.34	<1	90.22 ± 5.17	<1

Table 10: *vertebral column*

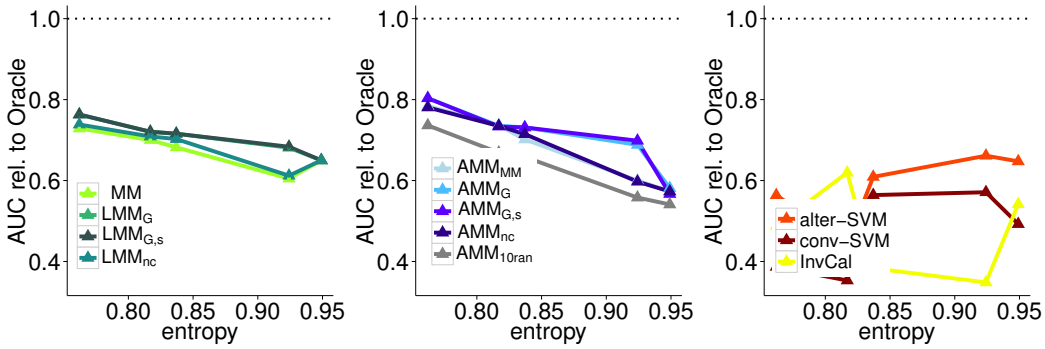
algorithm		2 bags AUC	2 bags time(s)	4 bags AUC	4 bags time(s)	8 bags AUC	8 bags time(s)	16 bags AUC	16 bags time(s)	32 bags AUC	32 bags time(s)
	EMM	57.91 ± 22.04	<1	59.05 ± 10.46	<1	51.43 ± 17.22	<1	45.39 ± 23.81	<1	61.30 ± 17.86	<1
	MM	77.45 ± 6.14	<1	78.97 ± 3.54	<1	79.85 ± 4.14	<1	82.74 ± 2.11	1	87.45 ± 3.57	1
	LMM_G	77.45 ± 6.14	3	78.34 ± 2.82	3	81.93 ± 3.81	3	87.52 ± 2.71	5	90.43 ± 3.20	6
	$LMM_{G,s}$	77.45 ± 6.14	9	78.34 ± 2.82	8	**83.87 ± 3.63**	9	**87.71 ± 2.56**	13	**91.06 ± 3.00**	14
	LMM_{nc}	77.45 ± 6.14	31	78.43 ± 2.74	31	80.02 ± 4.02	35	83.50 ± 2.46	54	88.10 ± 3.57	122
	InvCal	33.74 ± 24.95	4	36.46 ± 5.27	4	72.54 ± 5.79	4	61.89 ± 6.25	4	59.91 ± 8.79	4
AMM^{min}	AMM_{EMM}	**81.07 ± 8.12**	2	78.56 ± 8.66	2	90.56 ± 3.44	2	92.08 ± 1.78	2	93.14 ± 2.04	3
AMM^{min}	AMM_{MM}	75.64 ± 5.02	2	68.54 ± 4.90	2	87.10 ± 4.16	2	**92.66 ± 1.99**	3	93.50 ± 1.93	3
AMM^{min}	AMM_G	75.64 ± 5.02	6	69.27 ± 5.69	7	87.57 ± 4.48	8	92.45 ± 1.89	10	93.59 ± 1.83	11
AMM^{min}	$AMM_{G,s}$	75.64 ± 5.02	19	69.27 ± 5.69	22	87.86 ± 4.62	23	91.04 ± 3.82	30	92.97 ± 1.58	32
AMM^{min}	AMM_{nc}	75.64 ± 5.02	34	68.49 ± 4.86	35	88.33 ± 5.17	39	91.26 ± 3.98	59	**93.70 ± 2.09**	127
AMM^{min}	AMM_1	74.49 ± 6.08	1	68.66 ± 4.92	1	90.60 ± 3.18	2	92.41 ± 1.58	2	92.95 ± 1.75	2
AMM^{min}	AMM_{10ran}	76.42 ± 4.80	12	75.75 ± 5.07	16	**92.59 ± 0.22**	18	92.15 ± 1.44	15	92.46 ± 1.79	19
AMM^{max}	AMM_{EMM}	76.02 ± 12.70	4	78.42 ± 14.14	5	87.87 ± 1.94	5	87.88 ± 3.29	6	90.71 ± 2.79	8
AMM^{max}	AMM_{MM}	75.31 ± 13.69	5	**87.22 ± 3.13**	5	87.43 ± 2.59	6	88.85 ± 2.39	6	90.29 ± 2.47	9
AMM^{max}	AMM_G	75.31 ± 13.69	15	73.91 ± 16.06	17	87.89 ± 1.97	17	87.98 ± 3.27	21	90.29 ± 2.47	28
AMM^{max}	$AMM_{G,s}$	75.31 ± 13.69	44	67.48 ± 16.70	50	87.89 ± 1.97	51	87.98 ± 3.27	63	90.18 ± 3.26	82
AMM^{max}	AMM_{nc}	75.31 ± 13.69	43	82.97 ± 8.05	45	87.85 ± 2.00	49	88.91 ± 2.41	70	90.29 ± 2.47	144
AMM^{max}	AMM_1	77.35 ± 13.61	4	70.14 ± 17.19	5	84.17 ± 2.66	5	89.12 ± 2.31	6	90.94 ± 3.06	8
AMM^{max}	AMM_{10ran}	72.39 ± 14.33	36	82.49 ± 9.32	47	87.44 ± 1.52	47	85.79 ± 4.54	50	90.87 ± 2.53	69
SVM	alter-∝	40.88 ± 5.80	21	30.17 ± 7.47	23	68.26 ± 6.40	26	58.84 ± 21.21	33	37.17 ± 17.48	48
SVM	conv-∝	77.72 ± 6.23	3624	72.28 ± 8.88	2292	36.21 ± 8.38	2328	45.01 ± 14.91	2481	70.49 ± 5.59	2306
	Oracle	93.80 ± 1.06	<1	93.83 ± 1.67	<1	93.89 ± 1.89	<1	93.83 ± 1.62	<1	94.00 ± 1.42	<1

Table 11: *vote* (feature *physician-fee-freeze* was removed to make the problem harder)

algorithm		2 bags AUC	2 bags time(s)	4 bags AUC	4 bags time(s)	8 bags AUC	8 bags time(s)	16 bags AUC	16 bags time(s)	32 bags AUC	32 bags time(s)
	EMM	54.32 ± 8.79	<1	45.47 ± 15.63	<1	46.88 ± 6.06	1	55.20 ± 18.03	1	53.93 ± 10.59	1
	MM	94.56 ± 2.04	1	95.37 ± 2.62	2	95.65 ± 0.85	2	**96.33 ± 1.19**	2	96.74 ± 1.50	2
	LMM_G	94.56 ± 2.04	7	95.93 ± 2.47	8	95.87 ± 1.12	8	**96.41 ± 1.51**	9	**96.94 ± 1.67**	10
	$LMM_{G,s}$	94.56 ± 2.04	20	**96.03 ± 2.42**	22	**96.00 ± 1.18**	23	**96.38 ± 1.99**	25	96.81 ± 2.09	28
	LMM_{nc}	94.56 ± 2.04	28	95.83 ± 2.34	31	95.71 ± 0.92	43	96.23 ± 1.58	85	96.81 ± 1.50	234
	Invcal	**94.85 ± 1.71**	4	73.10 ± 2.21	4	77.86 ± 4.92	4	26.74 ± 6.82	4	79.77 ± 6.25	4
AMM^{min}	AMM_{EMM}	93.67 ± 1.84	2	95.04 ± 3.01	2	**96.18 ± 0.78**	2	96.43 ± 1.31	2	96.94 ± 1.62	3
AMM^{min}	AMM_{MM}	93.48 ± 2.31	2	95.12 ± 2.89	3	96.10 ± 0.82	3	96.15 ± 1.31	4	**97.30 ± 1.58**	4
AMM^{min}	AMM_G	93.48 ± 2.31	10	**95.61 ± 1.90**	12	95.92 ± 1.02	11	96.41 ± 1.12	13	**97.36 ± 1.47**	15
AMM^{min}	$AMM_{G,s}$	93.48 ± 2.31	29	94.87 ± 3.02	33	95.34 ± 0.98	35	96.11 ± 1.30	39	**97.36 ± 1.47**	46
AMM^{min}	AMM_{nc}	93.48 ± 2.31	32	95.38 ± 2.38	35	95.81 ± 1.01	46	96.03 ± 1.48	89	**97.38** ± 1.45	238
AMM^{min}	AMM_1	93.57 ± 1.99	2	94.32 ± 3.36	2	**96.25 ± 0.66**	2	96.17 ± 1.20	2	96.83 ± 1.42	2
AMM^{min}	AMM_{10ran}	**93.84 ± 2.23**	11	94.59 ± 3.56	11	95.85 ± 0.97	12	**96.63 ± 1.32**	15	96.66 ± 1.70	18
AMM^{max}	AMM_{EMM}	91.68 ± 0.81	11	94.97 ± 2.24	12	94.94 ± 1	13	95.83 ± 1.36	14	96.60 ± 1.31	15
AMM^{max}	AMM_{MM}	92.47 ± 0.38	12	93.43 ± 4.07	13	93.71 ± 1.34	14	95.40 ± 1.10	15	96.77 ± 1.31	17
AMM^{max}	AMM_G	92.47 ± 0.38	40	94.34 ± 2.65	34	94.03 ± 0.81	43	95.65 ± 1.70	48	96.45 ± 1.52	53
AMM^{max}	$AMM_{G,s}$	92.47 ± 0.38	124	94.22 ± 2.87	127	94.03 ± 0.81	132	96.01 ± 1.83	142	96.37 ± 1.39	160
AMM^{max}	AMM_{nc}	92.47 ± 0.38	65	94.96 ± 3.48	66	94.07 ± 0.78	78	95.14 ± 1.18	124	96.74 ± 1.31	275
AMM^{max}	AMM_1	91.60 ± 1.29	11	94.48 ± 2.14	12	94.34 ± 0.82	12	95.36 ± 1.56	13	96.54 ± 1.51	15
AMM^{max}	AMM_{10ran}	90.49 ± 2.02	101	94.59 ± 2.85	103	94.19 ± 0.73	104	95.73 ± 1.83	112	96.21 ± 1.67	128
SVM	alter-∝	51.58 ± 3.27	19	62.74 ± 4.27	21	60.88 ± 3.50	25	63.01 ± 9.51	33	41.87 ± 7.12	57
SVM	conv-∝	5.63 ± 2.03	1848	47.22 ± 4.92	1807	19.62 ± 5.91	1855	57.54 ± 11.22	1598	46.27 ± 9.48	1281
	Oracle	97.11 ± 1.31	<1	97.43 ± 2.25	<1	97.06 ± 0.87	<1	97.33 ± 1.38	<1	97.52 ± 1.49	<1

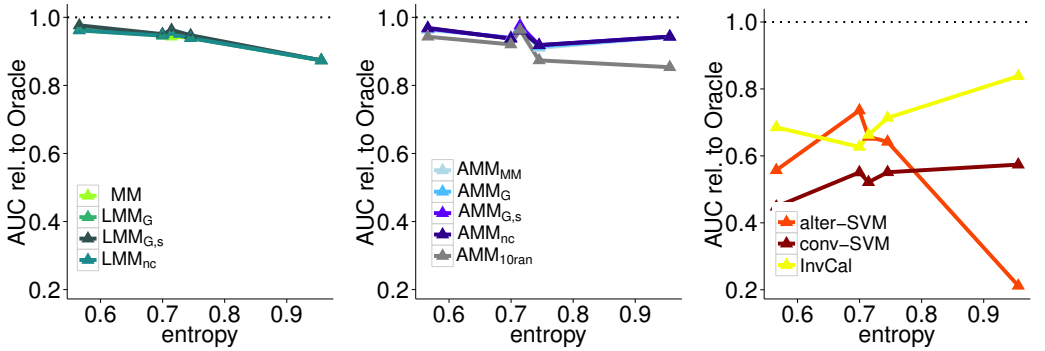
Table 12: *wine*

algorithm		2 bags		4 bags		8 bags		16 bags		32 bags	
		AUC	time(s)	AUC	time(s)	AUC	time(s)	AUC	time(s)	AUC	time(s)
	EMM	**70.38 ± 20.39**	<1	56.72 ± 29.85	<1	55.42 ± 20.70	<1	65.82 ± 21.45	<1	46.85 ± 16.71	<1
	MM	66.45 ± 5.42	1	82.41 ± 6.76	1	85.28 ± 4.80	1	90.35 ± 3.73	1	95.57 ± 2.45	1
	LMM_G	66.45 ± 5.42	4	89.72 ± 3.73	5	90.69 ± 5.30	5	94.09 ± 3.45	5	**97.74 ± 0.67**	6
	$LMM_{G,s}$	66.45 ± 4.412	13	**93.32 ± 2.94**	13	**92.68 ± 6.06**	14	**95.53 ± 2.40**	15	**97.69 ± 0.90**	19
	LMM_{nc}	66.45 ± 5.42	9	84.00 ± 5.48	11	86.30 ± 4.18	18	91.10 ± 4.52	40	96.28 ± 2.06	116
	Invcal	58.96 ± 5.77	6	81.38 ± 4.59	6	55.18 ± 9.59	6	63.07 ± 12.61	6	71.01 ± 18.19	6
AMM^{min}	AMM_{EMM}	80.27 ± 18.08	1	90.33 ± 8.87	1	91.46 ± 10.59	1	88.97 ± 6.26	1	88.34 ± 22.79	2
AMM^{min}	AMM_{MM}	61.84 ± 9.20	2	85.56 ± 7.20	1	88.70 ± 8.31	2	93.78 ± 9.12	2	98.66 ± 1.11	2
AMM^{min}	AMM_G	61.84 ± 9.20	6	93.06 ± 7.88	7	93.42 ± 8.24	7	96.09 ± 8.18	7	**99.33 ± 1.01**	9
AMM^{min}	$AMM_{G,s}$	61.84 ± 9.20	17	94.87 ± 5.68	18	93.00 ± 8.95	20	96.09 ± 8.18	21	**99.33 ± 1.01**	27
AMM^{min}	AMM_{nc}	61.84 ± 9.20	10	87.03 ± 3.93	13	88.23 ± 7.90	20	97.49 ± 5.06	43	**99.33 ± 1.01**	119
AMM^{min}	AMM_1	82.21 ± 11.39	<1	94.12 ± 6.34	1	99.60 ± 0.60	1	96.03 ± 7.57	1	97.03 ± 3.66	1
AMM^{min}	AMM_{10ran}	58.75 ± 31.30	4	**99.47 ± 0.68**	5	99.52 ± 0.45	6	**99.59 ± 0.54**	7	98.95 ± 1.66	10
AMM^{max}	AMM_{EMM}	74.23 ± 32.62	3	85.52 ± 17.48	4	99.67 ± 0.74	5	98.09 ± 3.09	6	92.00 ± 11.55	7
AMM^{max}	AMM_{MM}	88.23 ± 18.56	5	97.60 ± 2.40	4	87.42 ± 27.76	6	99.42 ± 0.79	7	98.61 ± 1.69	8
AMM^{max}	AMM_G	88.23 ± 18.56	15	88.41 ± 20	15	**100.00 ± 0.00** ↑	19	**99.63 ± 0.66**	20	98.61 ± 1.69	25
AMM^{max}	$AMM_{G,s}$	88.23 ± 18.56	44	79.11 ± 23.90	44	**100.00 ± 0.00** ↑	56	**99.63 ± 0.66**	59	98.61 ± 1.69	75
AMM^{max}	AMM_{nc}	88.23 ± 18.56	19	85.44 ± 19.04	21	86.17 ± 27.19	32	99.36 ± 0.74	56	98.61 ± 1.69	135
AMM^{max}	AMM_1	75.24 ± 21.10	3	80.45 ± 10.01	4	91.83 ± 14.63	5	91.79 ± 9.05	5	88.01 ± 9.78	7
AMM^{max}	AMM_{10ran}	**97.54 ± 1.55**	30	96.80 ± 3.94	32	99.46 ± 0.82	41	99.21 ± 0.79	47	98.54 ± 1.66	58
SVM	alter-∝	52.68 ± 2.54	14	36.53 ± 10.97	16	65.54 ± 2.26	19	29.15 ± 9.60	32	86.22 ± 11.93	44
SVM	conv-∝	54.31 ± 4.63	831	70.23 ± 6.58	794	52.88 ± 13.86	840	55.60 ± 11.29	659	11.58 ± 7.84	495
	Oracle	99.69 ± 0.52	<1	99.80 ± 0.44	<1	99.60 ± 0.43	<1	99.80 ± 0.44	<1	99.78 ± 0.33	<1



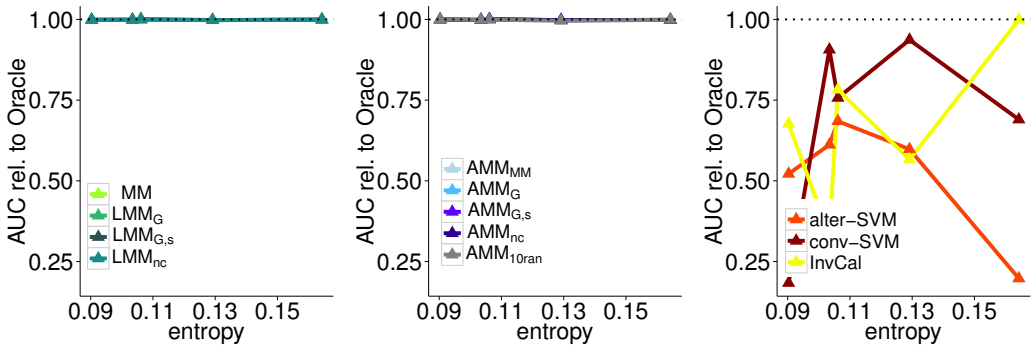
(a) (b) (c)

Figure 4: Relative AUC (wrt Oracle) vs entropy on *arrhythmia*



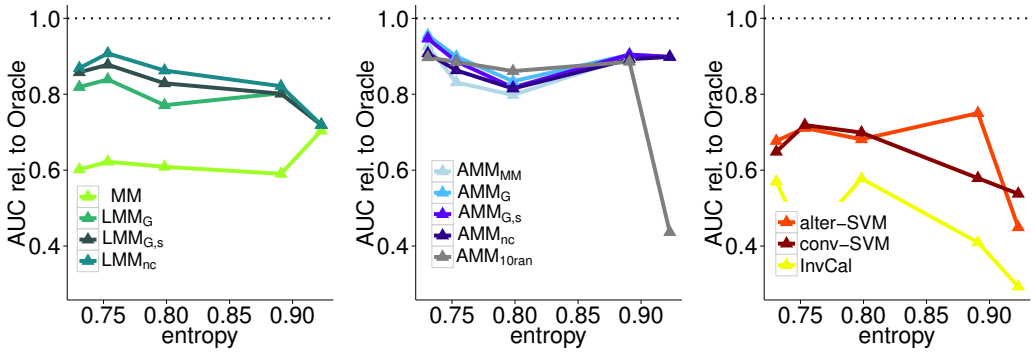
(a) (b) (c)

Figure 5: Relative AUC (wrt Oracle) vs entropy on *australian*



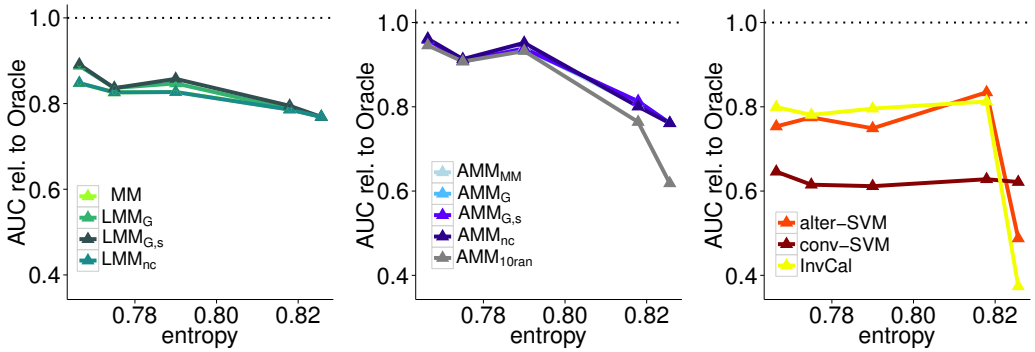
(a) (b) (c)

Figure 6: Relative AUC (wrt Oracle) vs entropy on *breastw*



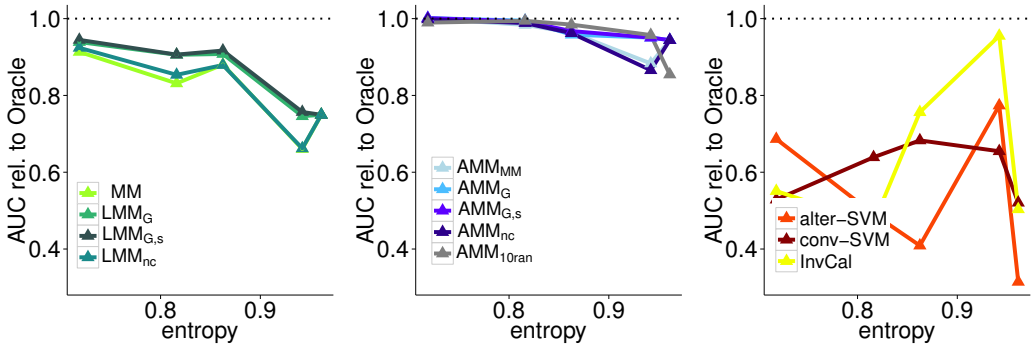
(a) (b) (c)

Figure 7: Relative AUC (wrt Oracle) vs entropy on *colic*



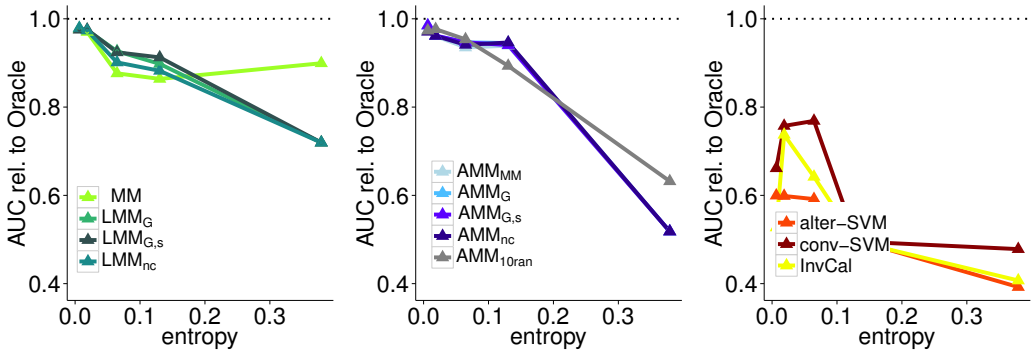
(a) (b) (c)

Figure 8: Relative AUC (wrt Oracle) vs entropy on *german*



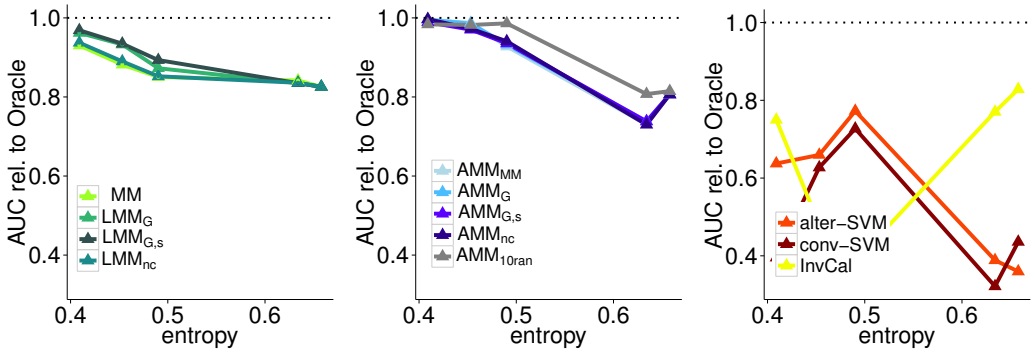
(a) (b) (c)

Figure 9: Relative AUC (wrt Oracle) vs entropy on *heart*



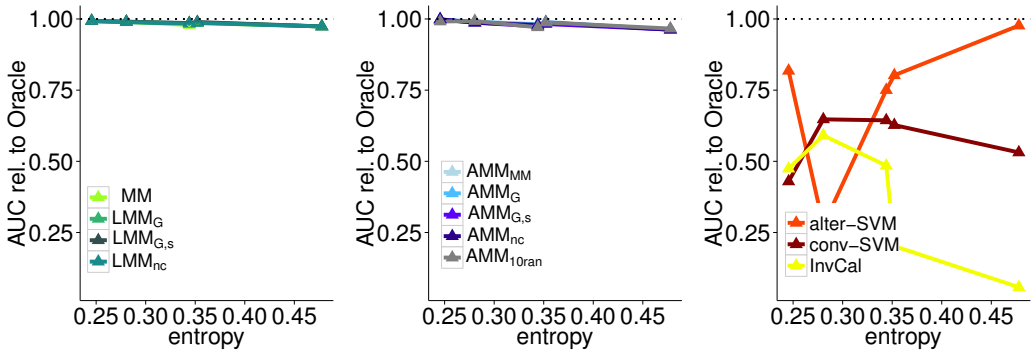
(a) (b) (c)

Figure 10: Relative AUC (wrt Oracle) vs entropy on *ionosphere*



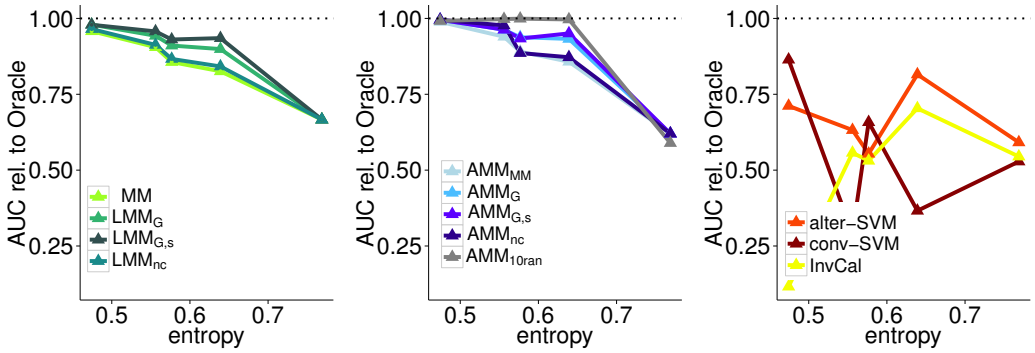
(a) (b) (c)

Figure 11: Relative AUC (wrt Oracle) vs entropy on *vertebral column*



(a) (b) (c)

Figure 12: Relative AUC (wrt Oracle) vs entropy on *vote*



(a) (b) (c)

Figure 13: Relative AUC (wrt Oracle) vs entropy on *wine*

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